Quantum Information, the Jones Polynomial and Khovanov Homology

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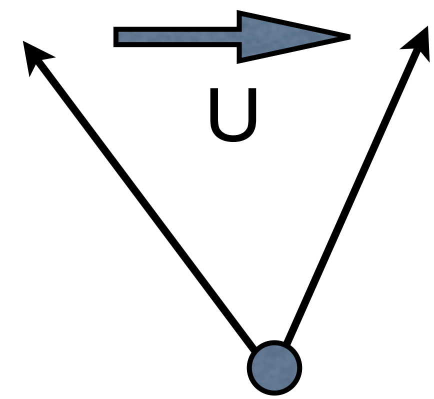
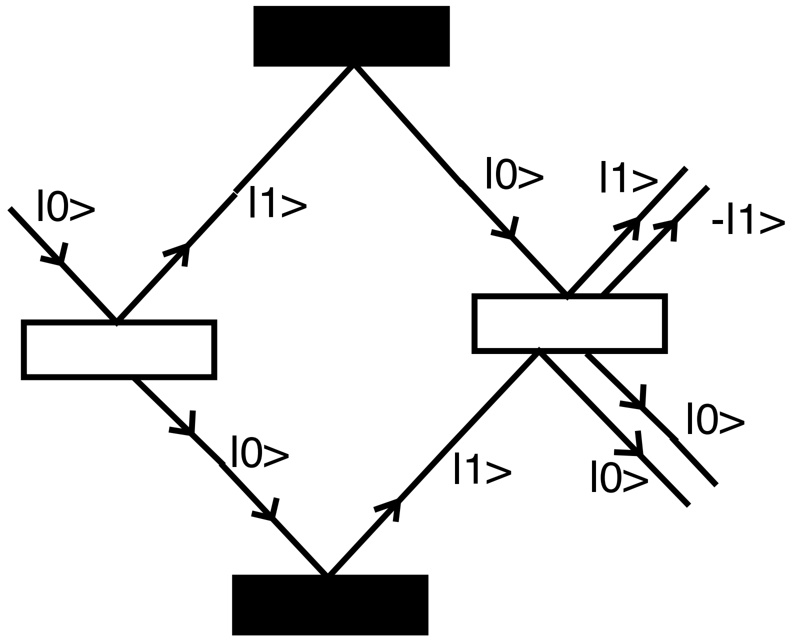
Quantum Mechanics in a Nutshell

0. A state of a physical system corresponds to a unit vector |S> in a complex vector space.

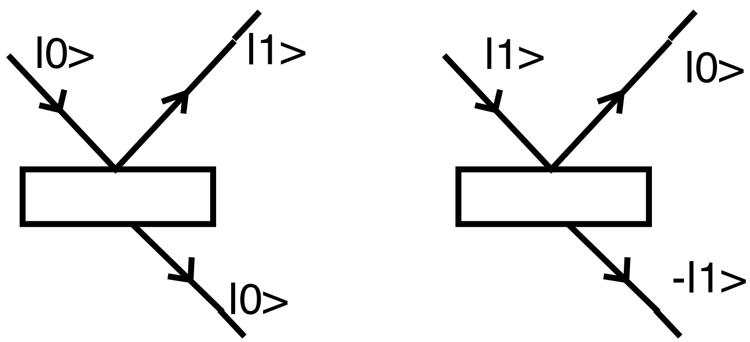
1. (measurement free) Physical processes are modeled by unitary transformations applied to the state vector: |S> -----> U|S>



2. If $|S\rangle = z_1|1\rangle + z_2|2\rangle + \ldots + z_n|n\rangle$ in a measurement basis {|1>,|2>,...,|n>}, then measurement of |S> yields |i> with probability $|z_i|$^2.



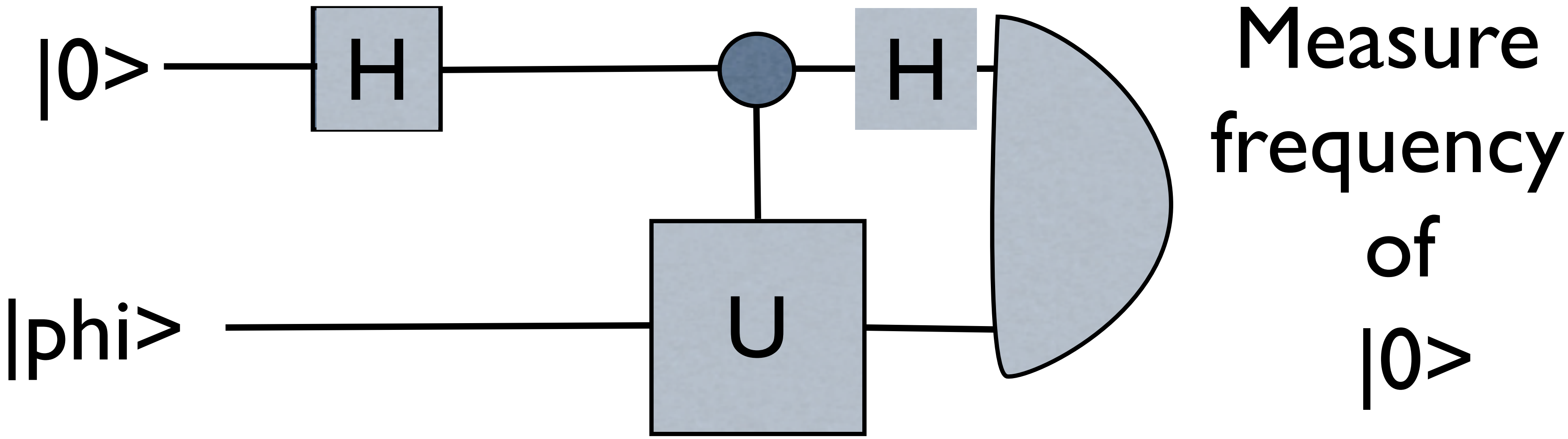
Mach-Zender Interferometer



$$H = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} / \mathrm{Sqrt}(2) \qquad M = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$HMH = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

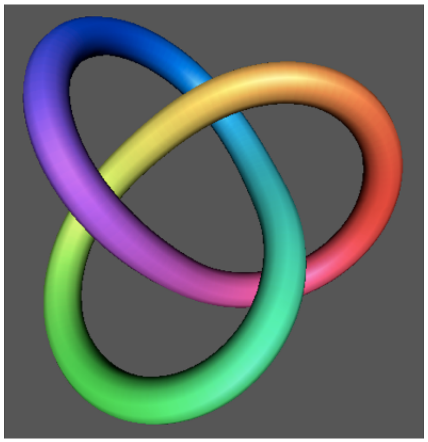
Hadamard Test - For Trace(U).



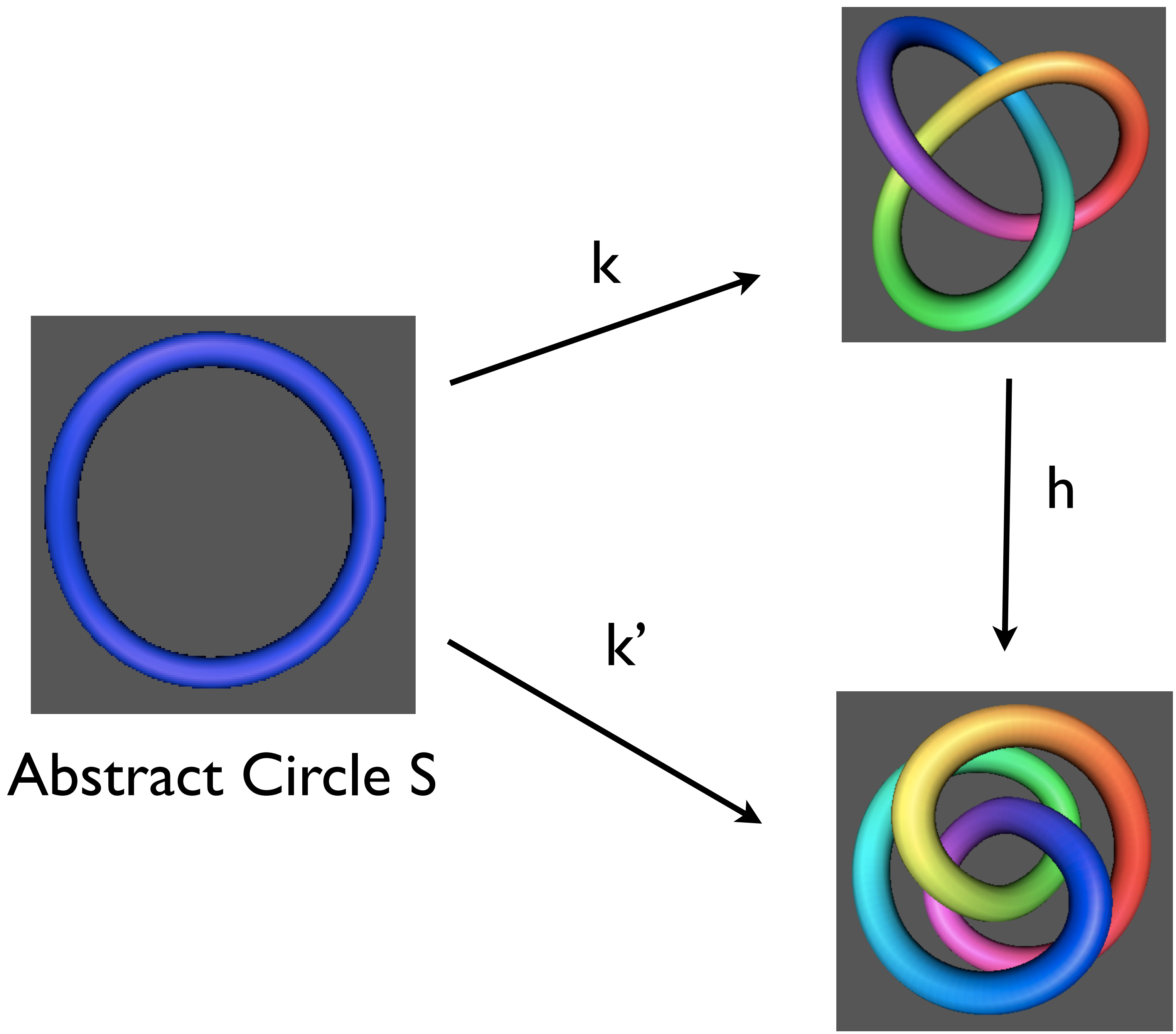
$$H = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} / \mathrm{Sqrt}(2)$$

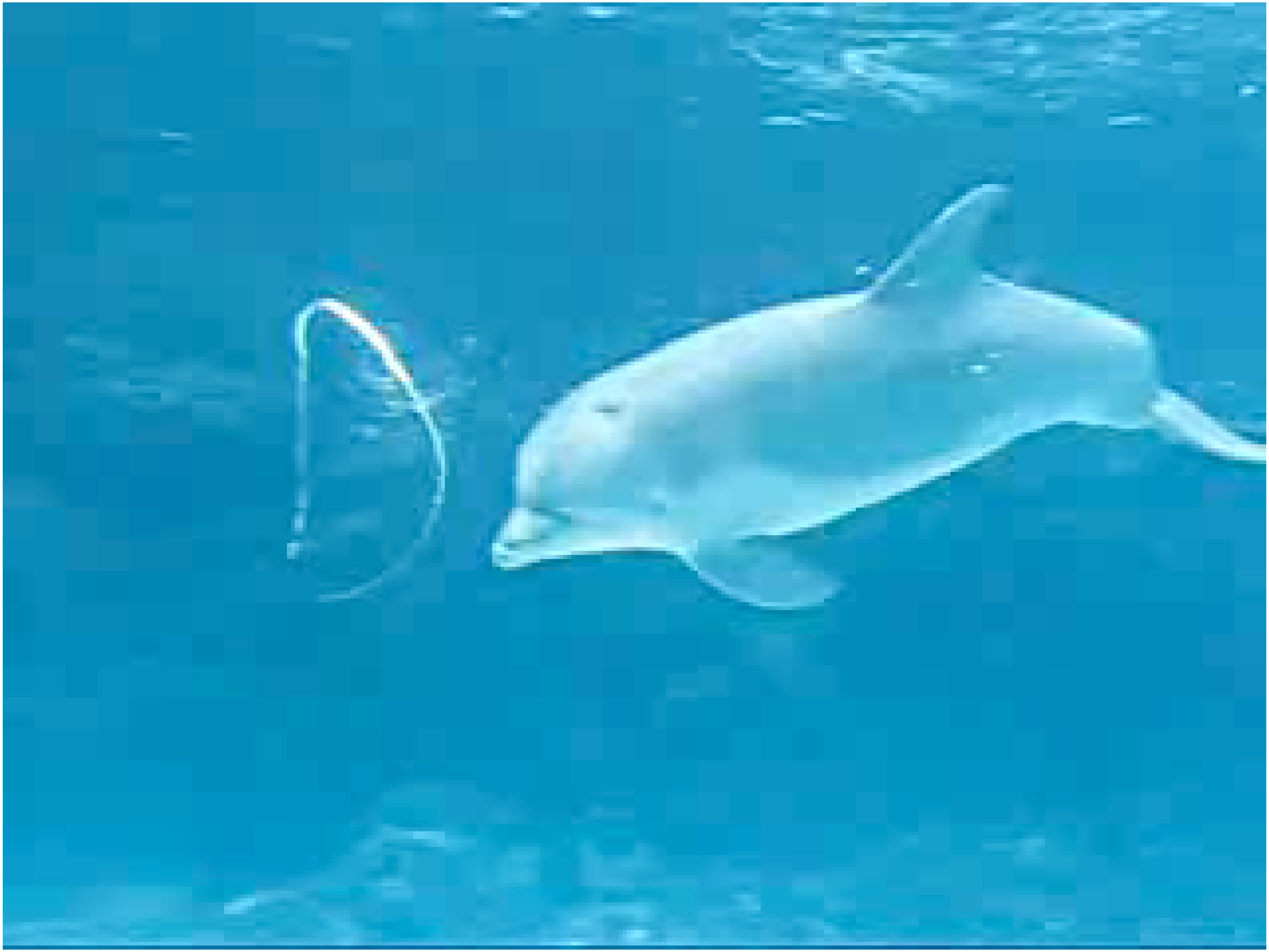
|0> occurs with probability
1/2 + Re[<phi|U|phi>]/2.

A knot is an embedding of a simple closed curve in three dimensional space.



Two knots K, L are equivalent if there is a homeomorphism h:R3 ----> R3 so that h(K) = L.


k
h
k’
Abstract Circle S



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Are Glueballs Knotted Closed Strings?

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May 29, 2006

Abstract

Glueballs have a natural interpretation as closed strings in Yang-Mills theory. Their stability requires that the string carries a nontrivial twist, or then it is knotted. Since a twist can be either left-handed or right-handed, this implies that the glueball spectrum must be degenerate. This degeneracy becomes consistent with experimental observations, when we identify the $\eta_L(1410)$ component of the $\eta(1440)$ pseudoscalar as a 0^{-+} glueball, degenerate in mass with the widely accepted 0^{++} glueball $f_0(1500)$. In addition of qualitative similarities, we find that these two states also share quantitative similarity in terms of equal production ratios, which we view as further evidence that their structures must be very similar. We explain how our string picture of glueballs can be obtained from Yang-Mills theory, by employing a decomposed gauge field. We also consider various experimental consequences of our proposal, including the interactions between glueballs and quarks and the possibility to employ glueballs as probes for extra dimensions: The coupling of strong interactions to higher dimensions seems to imply that absolute color confinement becomes lost.

Universal energy spectrum of tight knots and links in physics*

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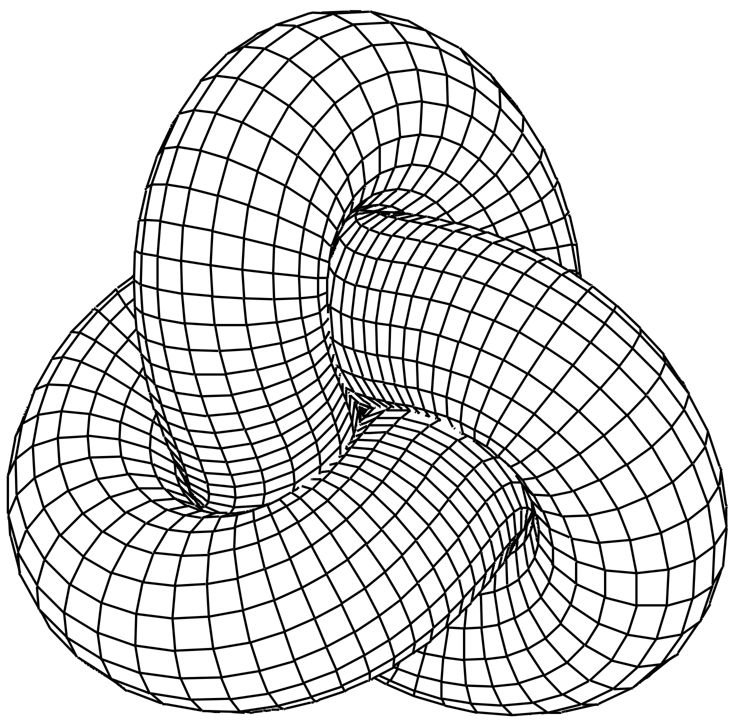
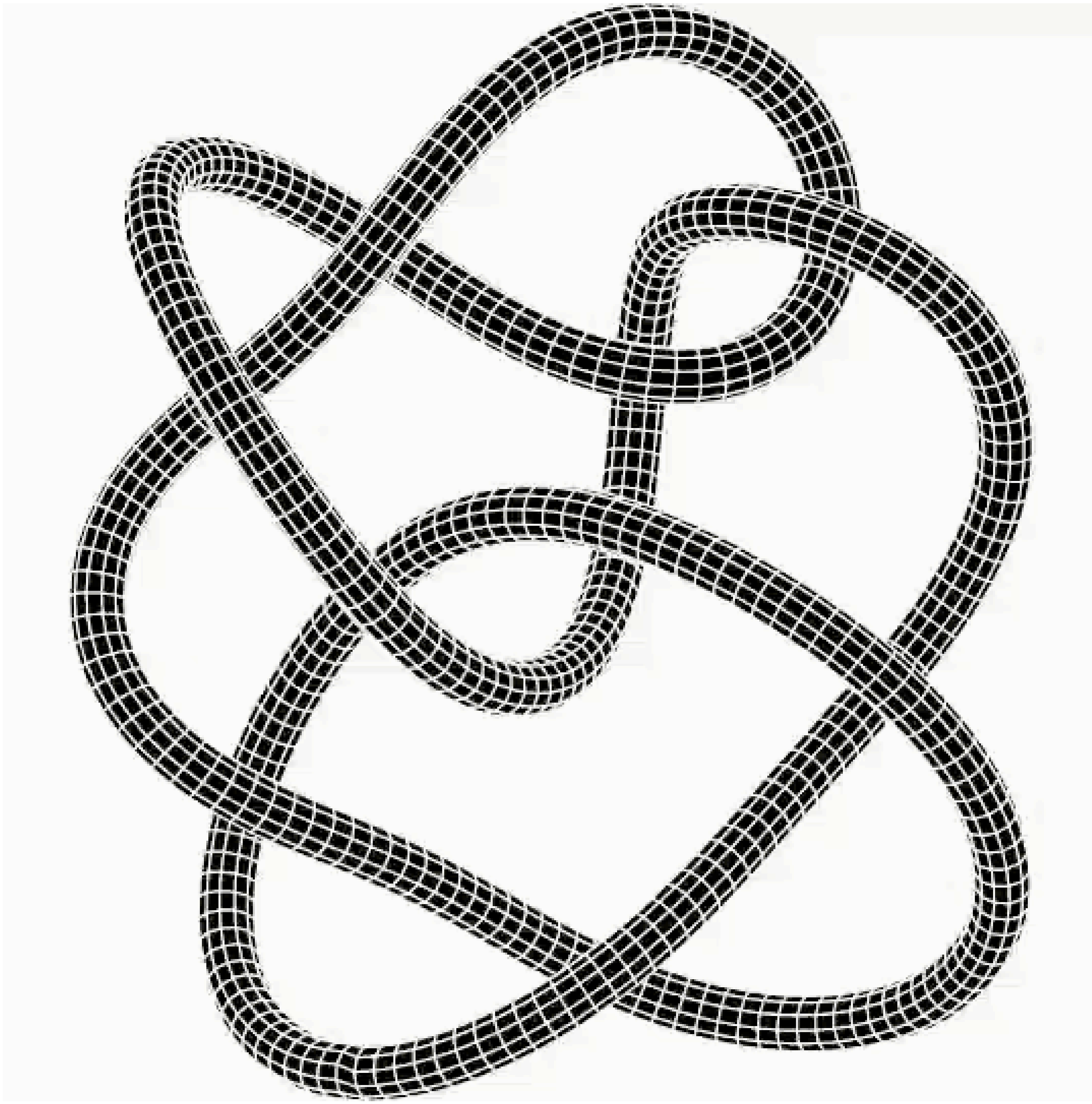
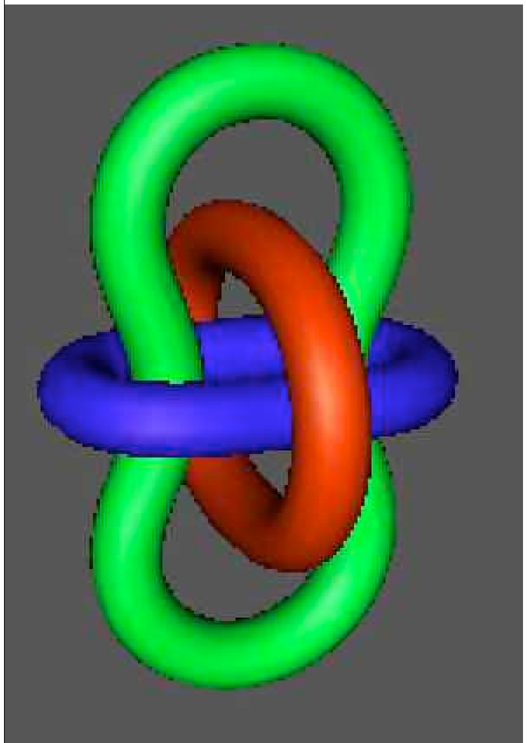
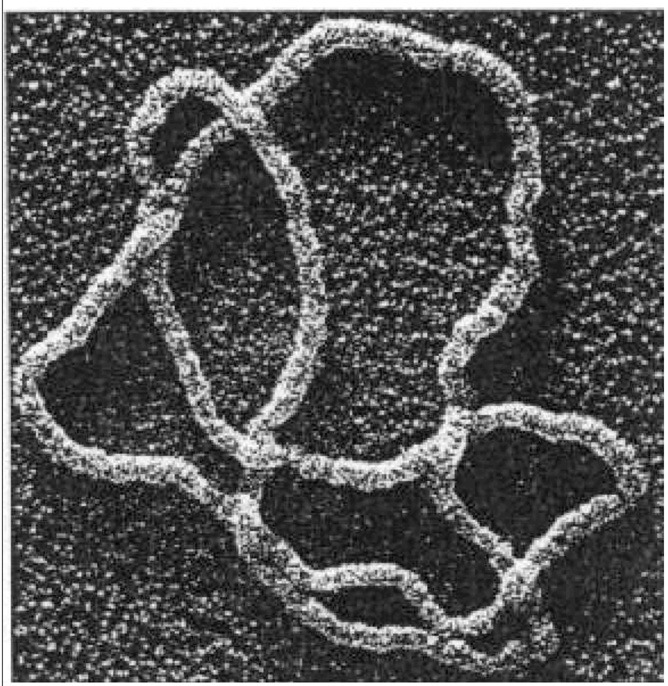
We argue that a systems of tightly knotted, linked, or braided flux tubes will have a universal mass-energy spectrum, since the length of fixed radius flux tubes depend only on the topology of the configuration. We motivate the discussion with plasma physics examples, then concentrate on the model of glueballs as knotted QCD flux tubes. Other applications will also be discussed.









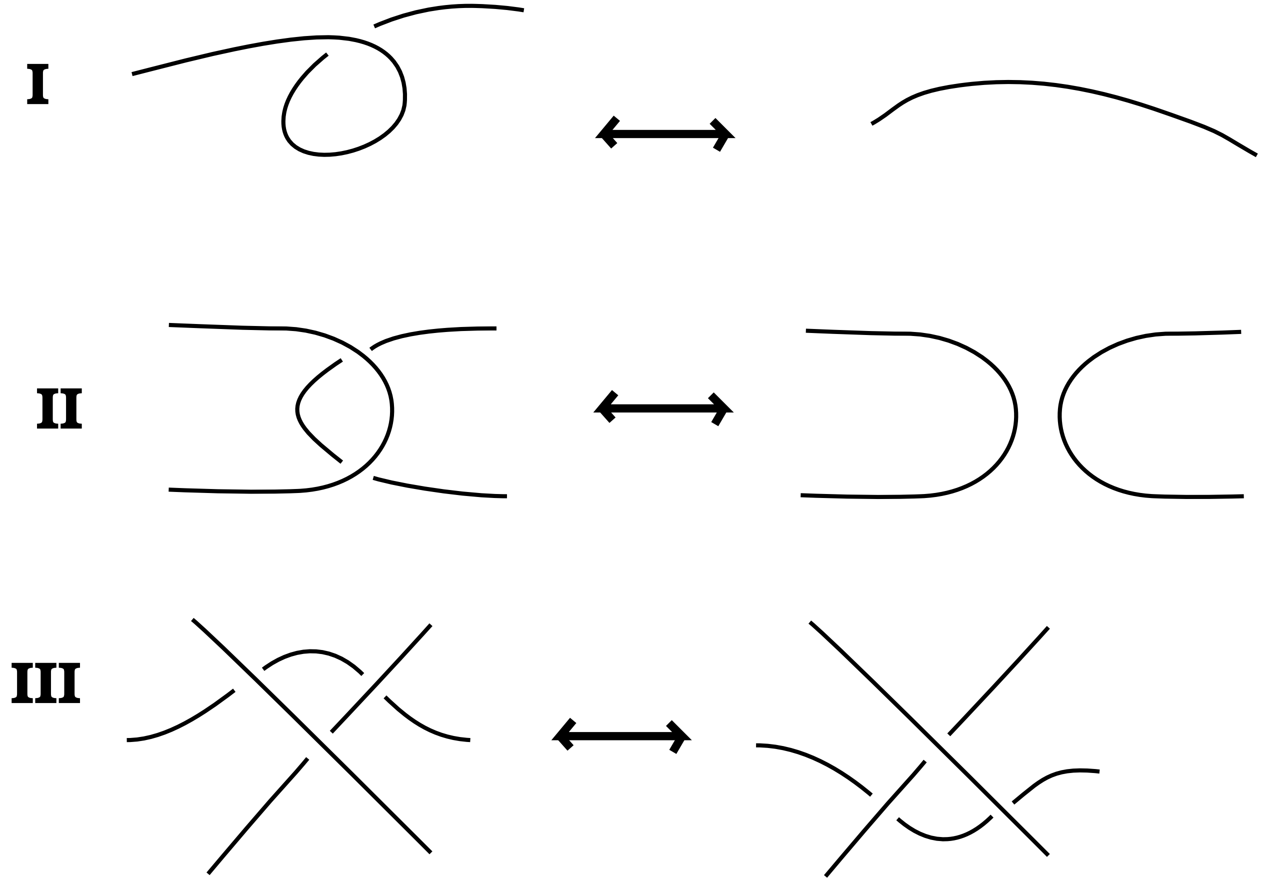
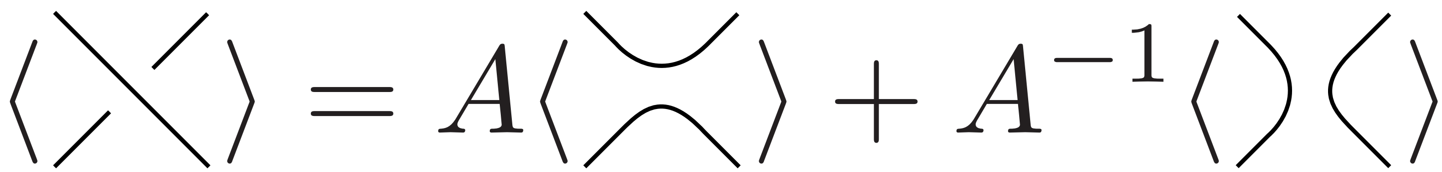


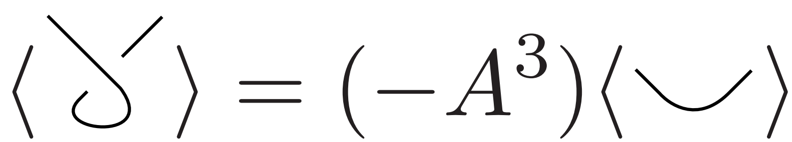
Figure 2 - The Reidemeister Moves.

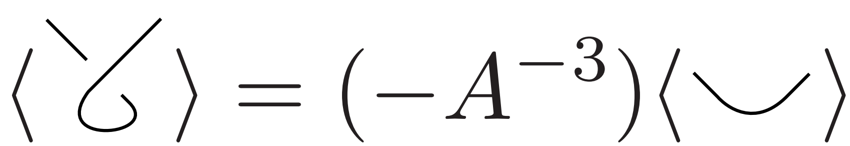
Reidemeister Moves reformulate knot theory in terms of graph combinatorics.

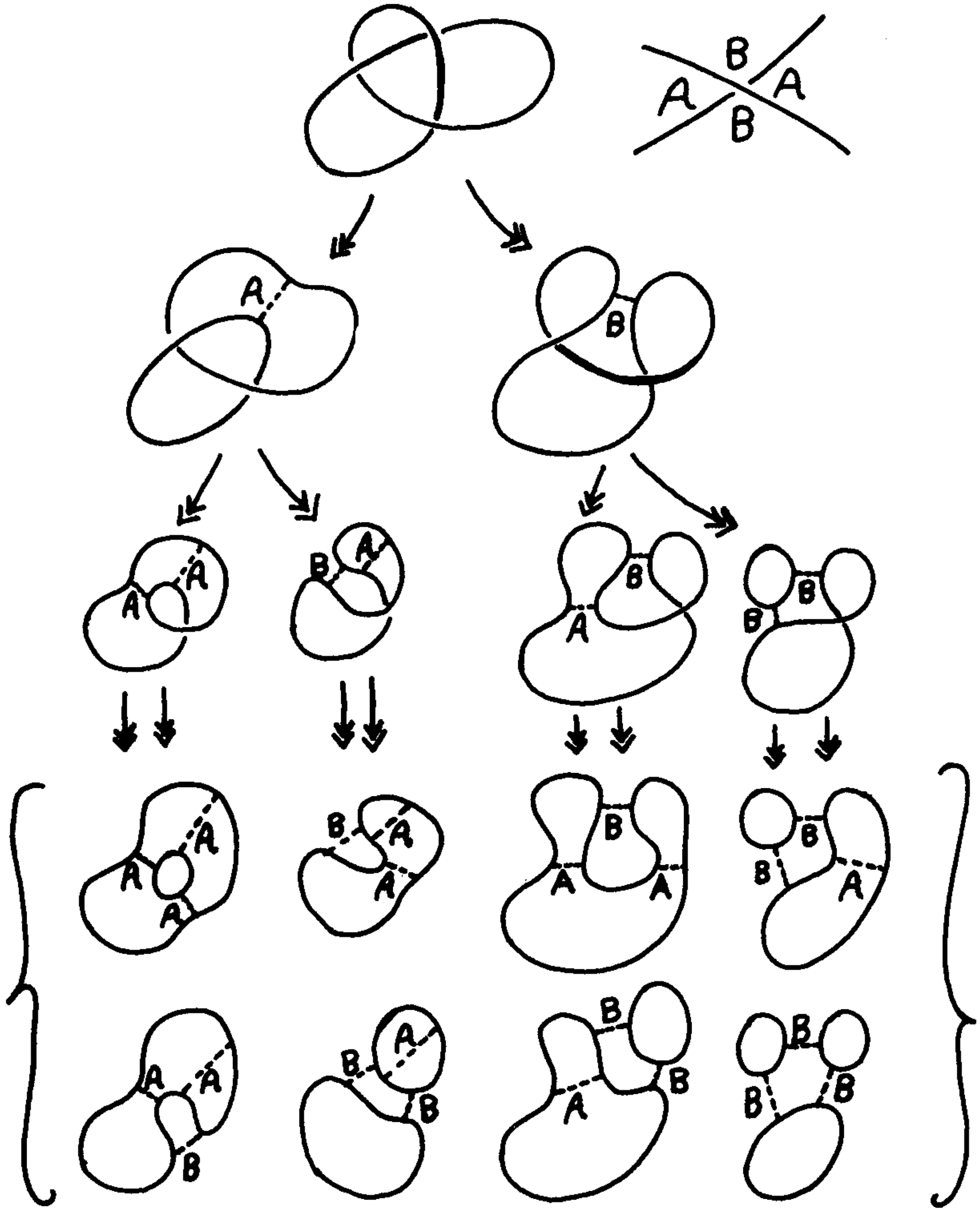
Bracket Polynomial Model for the Jones Polynomial



$$\langle K \bigcirc \rangle = (-A^{2} - A^{-2})\langle K \rangle$$

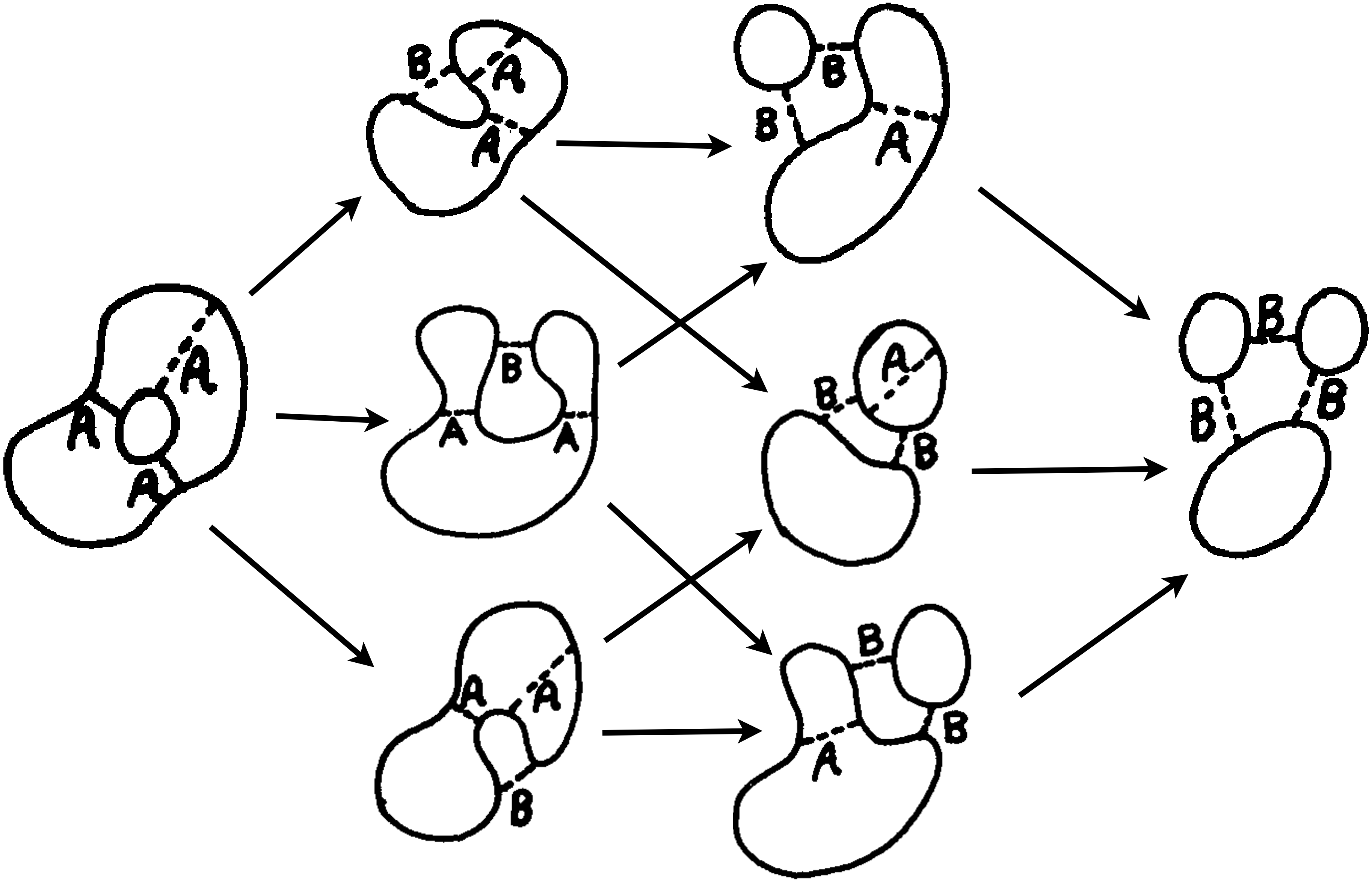






The form of the
expansion is the
same as
a loop expansion
of the Potts
model,
where the loops
are
boundaries of
regions of
constant spin.

The Khovanov Complex - A Cubical Organization of Bracket States



We will make a Hilbert space whose basis is a set of (enhanced) states of the bracket polynomial for a given knot diagram K.

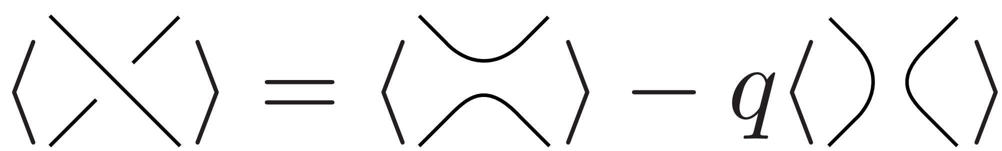
We associate a Hilbert space to an individual knot diagram.

Reformulating the Bracket

Let c(K) = number of crossings on link K.

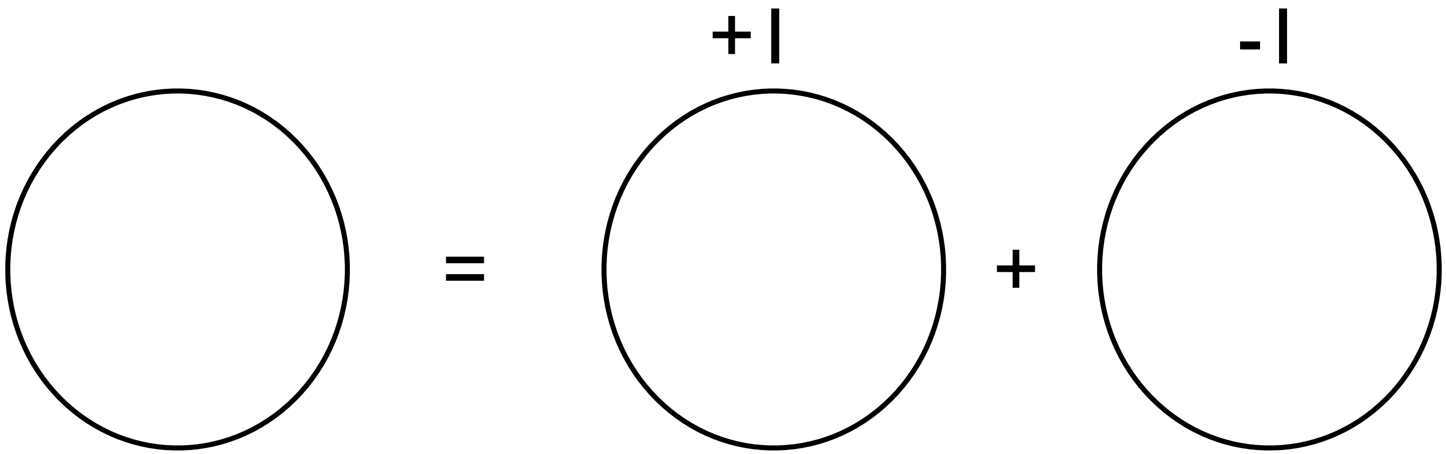
Form $A^{-c(K)}$<K> and replace A^{-2} by -q .

Then the skein relation for <K> will be replaced by:

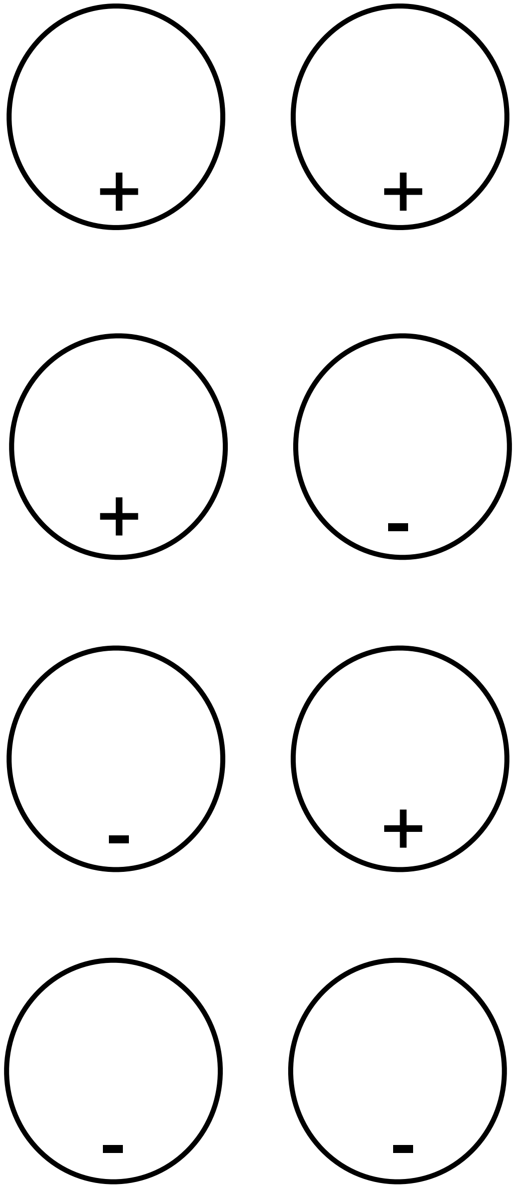


$$\langle \bigcirc \rangle = (q + q^{-1})$$

Use enhanced states by labeling each loop with +1 or -1.



$$\longleftrightarrow \qquad q + q^{-1}$$

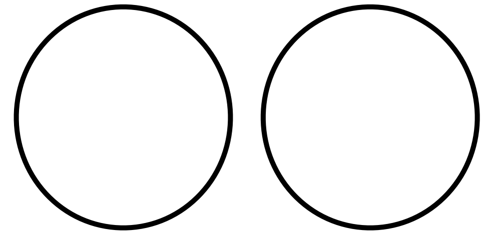


q^2

1

1

q^{-2}

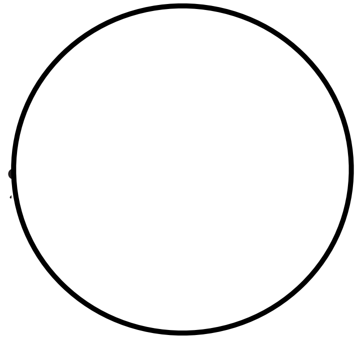


$(q + q^{-1})^2$

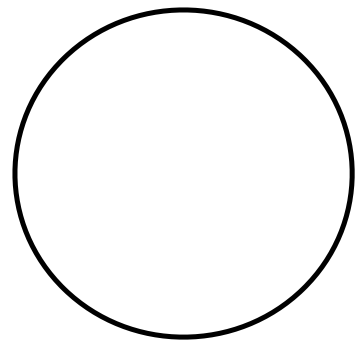
Enhanced States circumvent the binomial theorem.

Enhanced States

$$q^{-1} \Longleftrightarrow -1 \Longleftrightarrow X$$



$$q^{+1} \Longleftrightarrow +1 \Longleftrightarrow 1$$



For reasons that will soon become apparent, we let -1 be denoted by X and +1 be denoted by 1.

(The module V will be generated by 1 and X.)

Enhanced State Sum Formula for the Bracket

$$\langle K \rangle = \sum_{s} q^{j(s)}(-1)^{i(s)}$$

A Quantum Statistical Model for the Bracket Polynonmial.

Let C(K) denote a Hilbert space with basis |s> where s runs over the enhanced states of a knot or link diagram K.

We define a unitary transformation.

$$U : \mathcal{C}(K) \longrightarrow \mathcal{C}(K)$$

$$U|s\rangle = (-1)^{i(s)} q^{j(s)} |s\rangle$$

q is chosen on the unit circle in the complex plane.

<K> = Trace(U).

$$|\psi\rangle = \sum_s |s\rangle$$

Lemma. The evaluation of the bracket polynomial is given by the following formula

$$\langle K \rangle = \langle \psi | U | \psi \rangle .$$

This gives a new quantum algorithm for the Jones polynomial (via Hadamard Test).

Khovanov Homology - Jones Polynomial as an Euler Characteristic

Two key motivating ideas are involved in finding the Khovanov invariant. First of all, one would like to *categorify* a link polynomial such as $\langle K \rangle$. There are many meanings to the term categorify, but here the quest is to find a way to express the link polynomial as a *graded Euler characteristic* $\langle K \rangle = \chi_q \langle H(K) \rangle$ for some homology theory associated with $\langle K \rangle$.

We will formulate Khovanov Homology
in the context of our quantum statistical model for the bracket polynomial.

CATEGORIFICATION

View the next slide as a category.

The cubical shape of this category suggests making a homology theory.

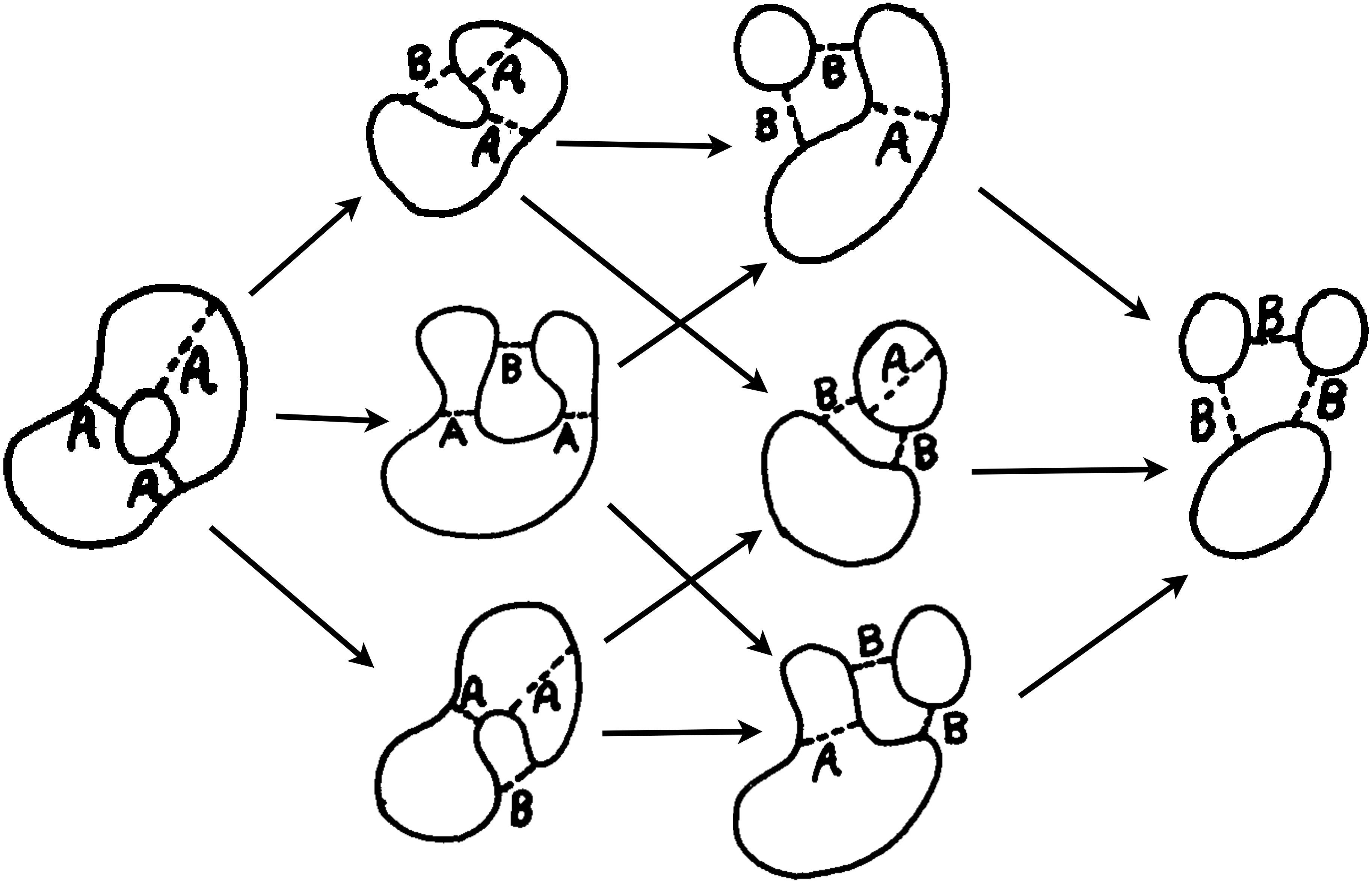
A Cubical Category Cube[2]

$$
\begin{tikzcd}
 & <AB> \arrow[dr] & \\
<AA> \arrow[ur] \arrow[dr] & & <BB> \\
 & <BA> \arrow[ur] &
\end{tikzcd}
$$

$$\partial <AA> = <BA> + <AB>$$

(mod 2 boundary)

The Khovanov Category

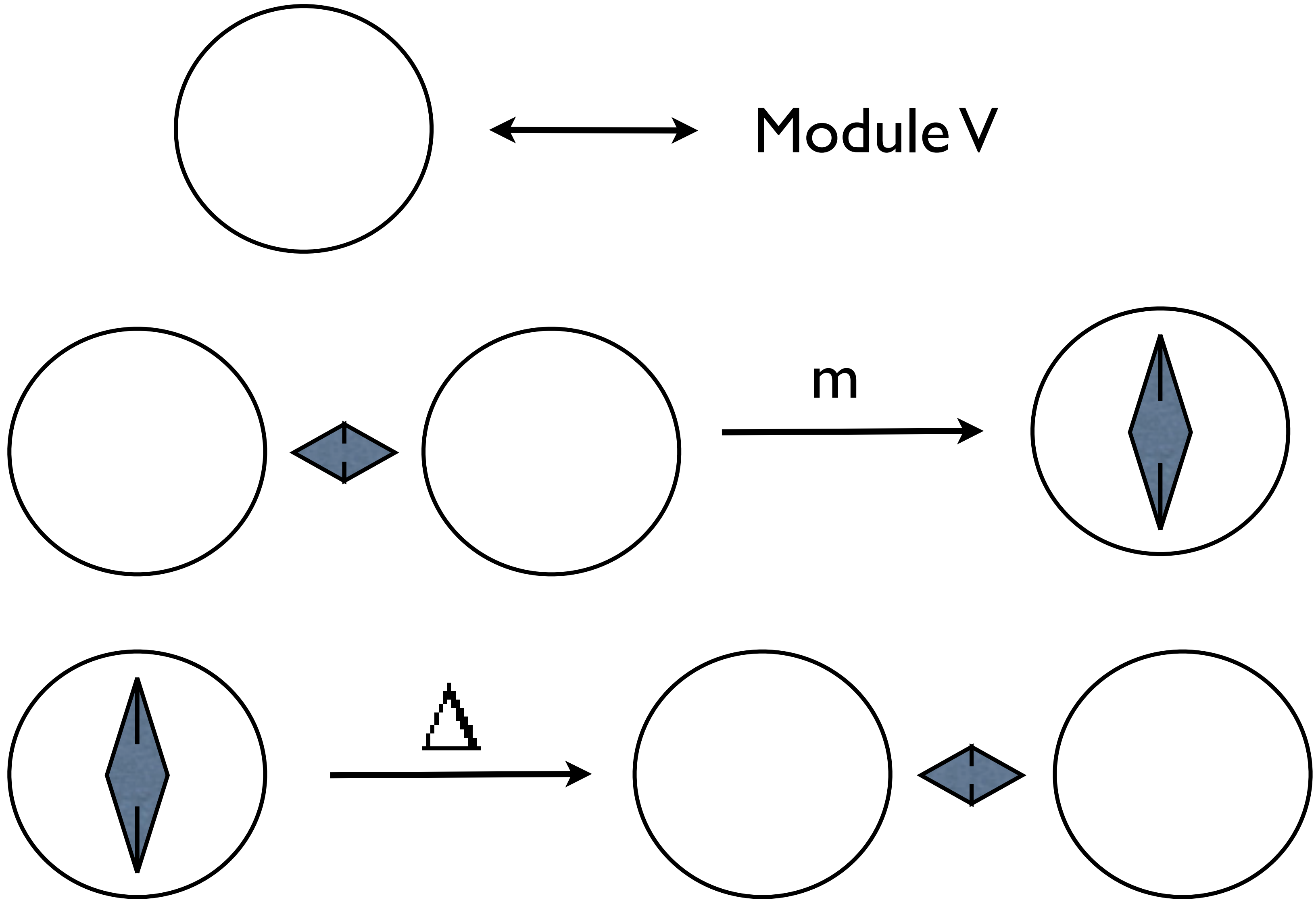


In order to make a non-trivial homology theory
we need a functor from this cubical category of states
to a module category.

Each state loop will map to a module V.

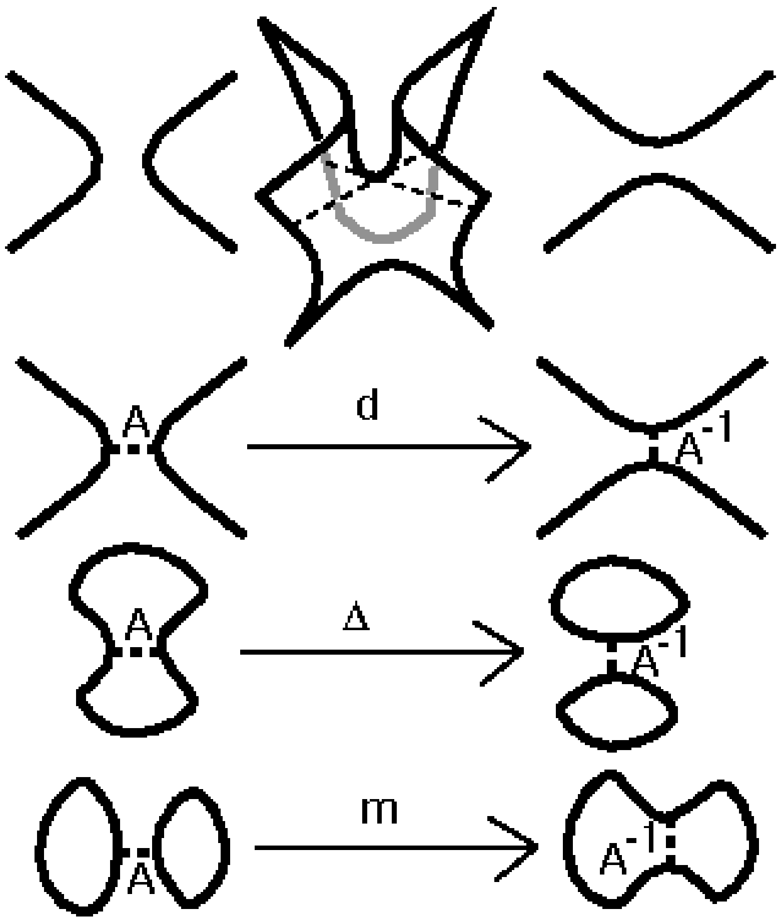
Unions of loops will map to
tenor products of this module.

The Functor from the cubical category to the module category demands multiplication and comultiplication in the module.



$$\partial(s) = \sum_{\tau} \partial_{\tau}(s)$$

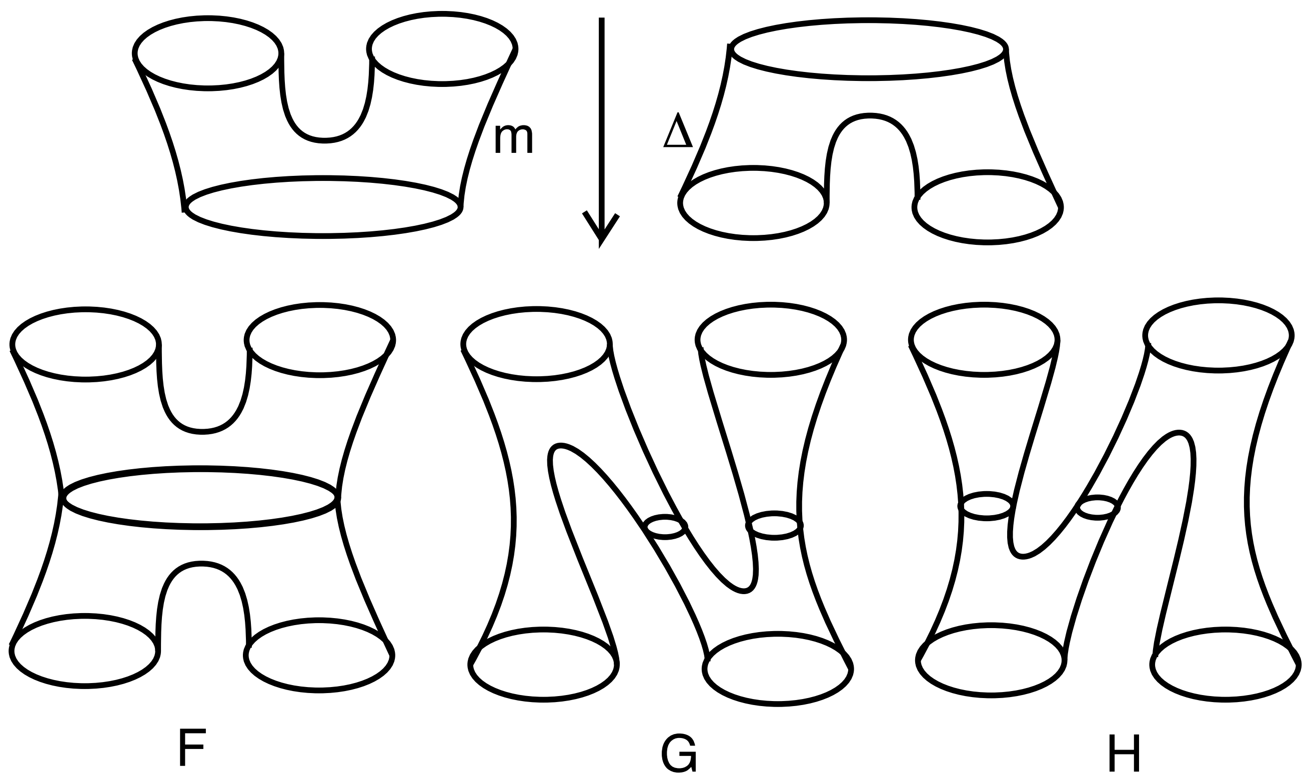
The boundary is a sum of partial differentials corresponding to resmoothings on the states.



Each state loop is a module.

A collection of state loops corresponds to a tensor product of these modules.

The commutation of the partial boundaries demands a structure of Frobenius algebra for the algebra associated to a state circle.



It turns out that one can take the algebra generated by 1 and X with

$X^2 = 0$ and

$$\Delta(X) = X \otimes X \text{ and } \Delta(1) = 1 \otimes X + X \otimes 1.$$

The chain complex is then generated by enhanced states with loop labels 1 and X.

Enhanced State Sum Formula for the Bracket

$$\langle K \rangle = \sum_s q^{j(s)}(-1)^{i(s)}$$

$$j(s) = n_B(s) + \lambda(s)$$

$i(s) = n_B(s)$ = number of B-smoothings in the state s.

$\lambda(s)$ = number of +1 loops minus number of -1 loops.

$$\langle K \rangle = \sum_{i\,,j} (-1)^i q^j dim(\mathcal{C}^{ij})$$

C^{ij} = module generated by enhanced states with $i = n_B$ and j as above.

$$\langle K \rangle = \sum_{i\,,j} (-1)^i q^j dim(\mathcal{C}^{ij})$$

The Khovanov differential acts in the form

$$\partial : \mathcal{C}^{ij} \longrightarrow \mathcal{C}^{i+1\,j}$$

(For j to be constant as i increases by 1,

$\lambda(s)$ decreases by 1.)

The differential increases the homological grading i by 1 and leaves fixed the quantum grading j.

Then

$$\langle K \rangle = \sum_j q^j \sum_i (-1)^i dim(\mathcal{C}^{ij}) = \sum_j q^j \chi(\mathcal{C}^{\bullet j})$$

$$\chi(H(\mathcal{C}^{\bullet j})) = \chi(\mathcal{C}^{\bullet j})$$

$$\langle K \rangle = \sum_j q^j \chi(H(\mathcal{C}^{\bullet j}))$$

$$\langle K \rangle = \sum_j q^j \chi(H(\mathcal{C}^{\bullet\, j}))$$

With $$U|s\rangle = (-1)^{i(s)} q^{j(s)} |s\rangle,$$

$$\partial : \mathcal{C}^{ij} \longrightarrow \mathcal{C}^{i+1\,j}$$

$$U\partial + \partial U = 0.$$

This means that the unitary transformation U acts on the homology so that

U: H(C(K)) -----> H(C(K))

Eigenspace Picture

$$\mathcal{C}^0 = \oplus_\lambda \mathcal{C}^0_\lambda$$

$$\mathcal{C}^\bullet_\lambda : \mathcal{C}^0_\lambda \longrightarrow \mathcal{C}^1_{-\lambda} \longrightarrow \mathcal{C}^2_{+\lambda} \longrightarrow \cdots \mathcal{C}^n_{(-1)^n\lambda}$$

$$\mathcal{C} = \oplus_\lambda \mathcal{C}^\bullet_\lambda$$

$$\langle\psi|U|\psi\rangle = \sum_\lambda \lambda\chi(H(C^\bullet_\lambda))$$

SUMMARY

We have interpreted the bracket polynomial as a quantum amplitude by making a Hilbert space C(K) whose basis is the collection of enhanced states of the bracket.

This space C(K) is naturally intepreted as the chain space for the Khovanov homology associated with the bracket polynomial.

$$\langle K \rangle = \langle \psi | U | \psi \rangle.$$

The homology and the unitary transformation U speak to one another via the formula

$$U\partial + \partial U = 0.$$

Questions

We have shown how Khovanov homology fits into the context of quantum information related to the Jones polynomial and how the polynomial is replaced in this context by a unitary transformation U on the Hilbert space of the model. This transformation U acts on the homology, and its eigenspaces give a natural decomposition of the homology that is related to the quantum amplitude corresponding to the Jones polynomial.

The states of the model are intensely combinatorial, related to the representation of the knot or link.

How can this formulation be used in quantum information theory and in statistical mechanics?!

Homework Problem -- Relate With:

Fivebranes and Knots

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Abstract

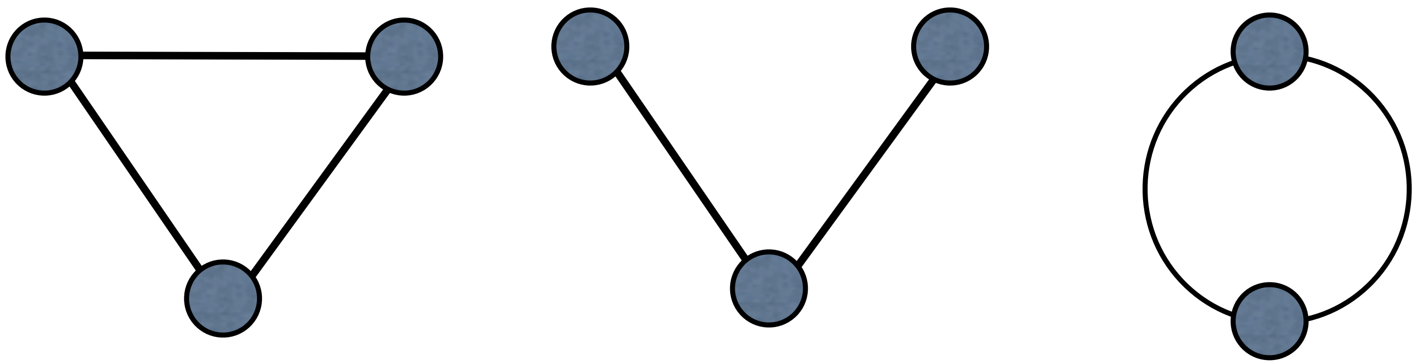
We develop an approach to Khovanov homology of knots via gauge theory (previous physics-based approches involved other descriptions of the relevant spaces of BPS states). The starting point is a system of D3-branes ending on an NS5-brane with a nonzero theta-angle. On the one hand, this system can be related to a Chern-Simons gauge theory on the boundary of the D3-brane worldvolume; on the other hand, it can be studied by standard techniques of S-duality and T-duality. Combining the two approaches leads to a new and manifestly invariant description of the Jones polynomial of knots, and its generalizations, and to a manifestly invariant description of Khovanov homology, in terms of certain elliptic partial differential equations in four and five dimensions.

The Dichromatic Polynomial and the Potts Model

Dichromatic Polynomial

$$Z[G](v,Q) = Z[G'](v,Q) + vZ[G''](v,Q)$$

$$Z[\bullet \sqcup G] = QZ[G].$$

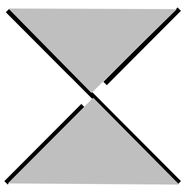
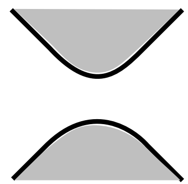
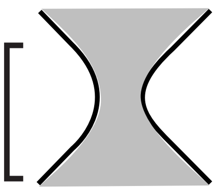


G

G’
Delete

G”
Contract

Precursor:

$Z[$  $] = Z[$  $] + vZ[$  $].$

delete contract

$$Z[R \sqcup K] = QZ[K].$$

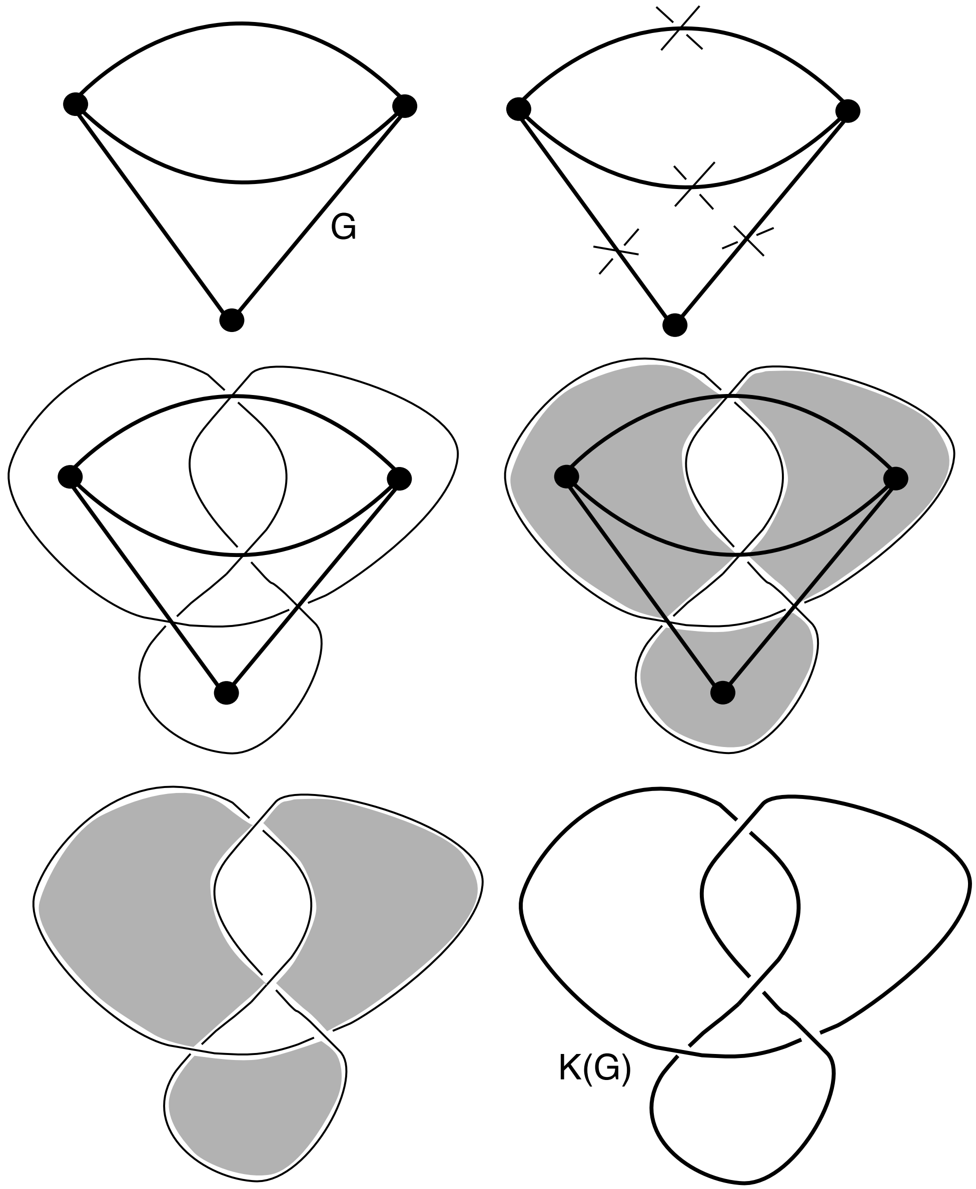


Figure 4: **Medial Graph, Checkerboard Graph and K(G)**

Partition Function

Recall that the partition function of a physical system has the form of the sum over all states s of the system the quantity

$$\sum \exp[(J/kT)E(s)]$$

where

J = +1 or -1 (ferromagnetic or antiferromagnetic models)

k = Boltzmann's constant

T = Temperature

E(s) = energy of the state s

Potts Model

In the Potts model, one has a graph G and assigns labels (spins, charges) to each node of the graph from a label set $\{1,2,\ldots,Q\}$.
A state s is such a labeling.
The energy E(s) is equal to the number of edges in the graph where the endpoints of the edge receive the same label.

For Q = 2, the Potts model is equivalent to the Ising model. The Ising model was shown by Osager to have a phase transition in the limit of square planar lattices (in the the 1940's).

The partition function $P_G(Q,T)$ for the Q-state Potts model on a graph G is given by the dichromatic polynomial

$$Z[G](v,Q)$$

where

$$v = e^{J\frac{1}{kT}} - 1$$

J = +1 or -1 (ferromagnetic or antiferromagnetic models)

k = Boltzmann's constant

T = Temperature

Let $K = J\frac{1}{kT}$

$$v = e^K - 1$$

$$P_G(Q,T) = Z[G](e^K - 1, Q)$$

$$P_G(Q,T) = \sum_{\sigma} e^{KE(\sigma)}$$

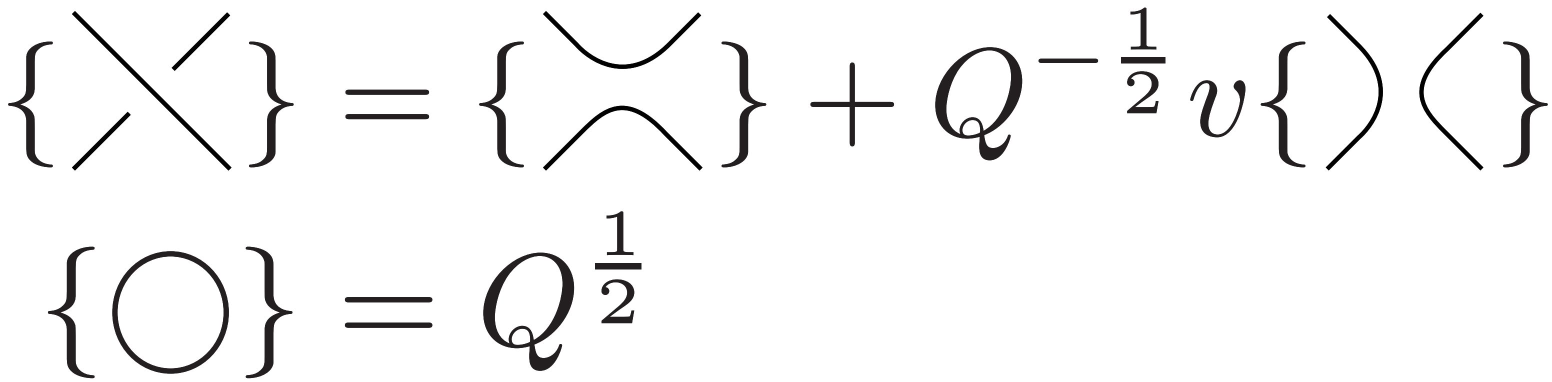
For planar graphs G we have

$$P_G(Q,T) = Q^{N/2}\{K(G)\}(Q, v = e^K - 1)$$

N = number of nodes of G.

Theorem: $Z[G](v,Q) = Q^{N/2}\{K(G)\}$

where K(G) is an alternating link associated with the medial graph of G and



For the Potts Model the critical point is at

$$Q^{-\frac{1}{2}}v = 1$$

To analyze Khovanov homology, we adopt a new bracket

$$[\times] = [\asymp] - q\rho[)(]$$

$$[\bigcirc] = q + q^{-1}$$

When rho = 1, we have the topological bracket in Khovanov form.

When

$$-q\rho = Q^{-\frac{1}{2}}v$$

$$q + q^{-1} = Q^{\frac{1}{2}}$$

we have the Potts model.

$$[K] = \sum_{s} (-\rho)^{n_B(s)} q^{j(s)}$$

$$[K] = \sum_{i,j} (-\rho)^{i} q^{j} dim(\mathcal{C}^{ij})$$

$$= \sum q^{j} \sum (-\rho)^{i} dim(\mathcal{C}^{ij}) = \sum q^{j} \chi_{\rho}(\mathcal{C}^{\bullet j})$$

where

$$\chi_{\rho}(\mathcal{C}^{\bullet j}) = \sum_{i} (-\rho)^{i} dim(\mathcal{C}^{ij}).$$

$$[K](q, \rho = 1) = \sum_j q^j \chi_\rho(\mathcal{C}^{\bullet\, j}) = \sum_j q^j \chi(H(\mathcal{C}^{\bullet\, j}))$$

Away from rho=1, one can ask what is the influence of the Khovanov homology on the coefficients in the expansion of

$$[K](q, \rho)$$

and corresponding questions about the Potts model.

Tracking Potts

$$-q\rho = Q^{-\frac{1}{2}}v$$

$$q + q^{-1} = Q^{\frac{1}{2}}$$

whence

$$q^2 - \sqrt{Q}q + 1 = 0.$$

$$q = \frac{\sqrt{Q} \pm \sqrt{Q-4}}{2}$$

At criticality Potts meets Khovanov at four colors and imaginary temperature!

$$\frac{1}{q} = \frac{\sqrt{Q} \mp \sqrt{Q-4}}{2}.$$

Criticality: $-\rho q = 1$

$$\rho = -\frac{1}{q} = \frac{-\sqrt{Q} \pm \sqrt{Q-4}}{2}.$$

Suppose that $\rho = 1.$

Then $2 = -\sqrt{Q} \pm \sqrt{Q-4}.$

So $4 - Q = \mp\sqrt{Q}\sqrt{Q-4}.$

And need Q = 4 and $e^K = -1$.

Now consider rho = 1 without insisting on criticality.

$$1 = -v/(q\sqrt{Q})$$

$$\rho = -\frac{v}{\sqrt{Q}q} = v(\frac{-1 \pm \sqrt{1 - 4/Q}}{2})$$

$$v = -q\sqrt{Q} = \frac{-Q \mp \sqrt{Q}\sqrt{Q-4}}{2}.$$

$$e^K = 1 + v = \frac{2 - Q \mp \sqrt{Q}\sqrt{Q-4}}{2}.$$

For $Q = 2$ we have $e^K = \pm i$.

For Q = 3, $e^K = \frac{-1 \pm \sqrt{3}i}{2}$.

For $Q = 4$ we have $e^K = -1$.

For Q >4, e^K is real and negative.

Thus we get complex temperature values in all cases where the coefficients of the Potts model are given directly in terms of Euler characteristics from Khovanov homology.