Bulk deformations of open topological string theory

Michael Kay

Based on work with N. Carqueville (1104.5438)

The Aim

At genus zero, open topological string theory consists of

$$\langle\psi_a(t_0),\,\psi_b(t_1)Pe^{\int_{t_1}^{t_2}\sum_k u_k\psi_k^{(1)}}\psi_c(t_2)\rangle_{Disk}$$

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Restrict to affine Landau Ginzburg models: aim achieved via **closed to open** string field theory

A_∞/L_∞ preliminaries

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In particular ∂ determined by $\partial^1_m \in \mathrm{Hom}(A[1]^{\otimes m}, A[1])$, more precisely we choose

$$\mathrm{Coder}(T_A) \cong \bigoplus_{n\geq 0} \mathrm{Hom}(A[1]^{\otimes n}, A[1])$$

A_∞/L_∞ preliminaries II

$$\langle\psi_{a_0}(t_0),\psi_{a_1}(t_1)\int\psi^{(1)}_{a_2}\dots\int\psi^{(1)}_{a_{n-1}}\psi_{a_n}(t_n)\rangle=$$

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Let $\widetilde{\partial} \in \mathrm{Coder}(T_H)$, with

$$\widetilde{\partial}^1_n := r_n = s \circ m_n \circ (s^{-1})^{\otimes n}$$

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then

BRST Ward ID

Ward identity for BRST operator $Q \iff \widetilde{\partial}^2 = 0,$

[*Costello* '06; *Herbst*, *Lerche*, *Lazaroiu* '04]

i.e. <u>$(H, \widetilde{\partial})$ is an A_∞-algebra</u>. In fact it is <u>minimal</u>: $r_0 = r_1 = 0$

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Let (A, ∂) be a strong A_∞-algebra with r_1-cohomology H. There is a unique, up to isomorphism, coalgebra morphism $F \in \mathrm{Hom}(T_H, T_A)$ and minimal A_∞-structure $\widetilde{\partial} \in \mathrm{Coder}(T_H)$.

Trees determined by G:

$$r_1 G + G r_1 = Id_A - \pi_H$$

Minimal Model theorem II

Problem: [Cyclicity] Ward identities for conjugate to $Q \Longrightarrow$

$$\langle \psi_{a_0}, r_n(\psi_{a_1} \otimes \cdots \otimes \psi_{a_n}) \rangle = \pm \langle \psi_{a_n}, r_n(\psi_{a_0} \otimes \cdots \otimes \psi_{a_{n-1}}) \rangle$$

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Resolution: Algorithmically, via non-commutative symplectic geometry$_{[Carqueville\,'09; Kontsevich, Soibelman\,'06]}$

Bulk Deformations Preliminaries

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Bulk Deformation

$$\widetilde{\partial} \mapsto \widetilde{\partial} + \widetilde{\delta}$$

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First order solutions are $\in H_{[\widetilde{\partial}, \cdot]}(\mathrm{Coder}(T_H)) =: HH^\bullet(H, \widetilde{\partial})$.

B-twisted Affine LG models

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TFT data

$$H_c = Jac(W) = \mathbb{C}[x_1, \ldots, x_n]/(\partial_1 W, \ldots, \partial_N W)$$
$$H_o = H_{[D,\cdot]}(A)$$

Closed/Open TST

Closed TST is the L_∞-minimal model of

closed off-shell (SFT)

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where $\partial^1 = \partial^1_1 + \partial^1_2$ with $\partial^1_1 = [D, \cdot]$ and $\partial^1_2 =$ matrix multiplication

Closed SFT to Open TST

Strategy: map closed deformation problem to the open sector:

$$S : (T_{poly}, ([-W, \cdot]_{SN}, [\cdot, \cdot]_{SN})) \to (\mathrm{Coder}(T_H), ([\widetilde{\partial}, \cdot], [\cdot, \cdot]))$$

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Given pure bulk deformation γ

$$\widetilde{\delta} = \sum_{k \geq 1} \frac{1}{n!} S^1_n(\gamma^{\wedge n})$$

is a **bulk induced** deformation

Work way backwards

Translation [N. Carqueville, M.K.]

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Cotrace [N. Carqueville, M.K.]

$$\mathrm{cotr} : (\mathrm{Coder}(T_R), ([\widehat{\partial}_{02}, \cdot], [\cdot, \cdot])) \to (\mathrm{Coder}(T_A), ([\partial_{02}, \cdot], [\cdot, \cdot]))$$

generalisation of Morita equivalence

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For $W \neq 0$, K is still L_∞-quasi-isomorphism

The Missing Ingredient

Off-shell (SFT) Kapustin-Li pairing [N. Carqueville, M.K.]:

$$(\mathcal{C}^2_{\mathrm{cl}}(B_A), L_{Q_{1,2}})^* \longrightarrow (\mathcal{C}^0(B_A), L_{Q_{1,2}})^* \longrightarrow (C^{\lambda}(A), b_{1,2})$$

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$$\xrightarrow{\mathrm{str}} (\mathsf{CC}_\bullet(R), \widehat{b}_{0,2} + uB) \xrightarrow{\mathsf{HKR}} (\Omega^\bullet(\mathbb{C}^N), dW \wedge (\,\cdot\,) + ud) \xrightarrow{\rho} (\mathbb{C}, 0)$$

Deformation Retraction: OSFT to OTST

$(T_H, \widetilde{\partial})$ is a deformation retract of (T_A, ∂)

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with

$$\Delta U = \frac{1}{2}(U \otimes (1 + F\bar{F}) + (1 + F\bar{F}) \otimes U)\Delta\,,$$
$$U_1^1 = G\,,$$
$$U_n^1 = -G\partial_2^1 U_n^2 \text{ for } n \geqslant 2\,,$$
$$\bar{F}_1^1 = \pi_H\,,$$
$$\bar{F}_n^1 = -\pi_H([\partial, U])_n^1 = -\pi_H \partial_2^1 U_n^2\,.$$

Deformation Retraction: OSFT to OTST II

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$\Longrightarrow$

$$\delta \mapsto \widetilde{\delta} = \bar{F}(1 - \delta U)^{-1} \delta F$$