Omega backgrounds and generalized holomorphic anomaly equation

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Minxin & A.K. arXiv 1009.1126 and Kashani-Poor, Minxin & A.K. June 2011

- Ω-background
- The generalized holomorphic anomaly
- Direct integration of the generalized holomorphic anomaly
- Applications: Topological Strings/ Gauge Theory
- Conclusions

What are Ω-backgrounds?

Formally they define a refinement of the N=2 topological string or SYM partition function:

$$\log Z(a,\epsilon_1,\epsilon_2)=\sum_{i,j=0}^{\infty}(\epsilon_1+\epsilon_2)^i(\epsilon_1\epsilon_2)^{j-1}F^{(\frac{i}{2},j)}(a)$$

Limits:

1.) For $\epsilon_1=-\epsilon_2=ig_s$ it becomes the usual genus exp.

$$\log Z(a,ig_s,-ig_s)=\sum_{g=0}^{\infty}g_s^{2g-2}F^{(0,g)}(a)\ .$$

Here g_s the string coupling, a denoted (vector) moduli.

2.) $\epsilon_1 = 0$, $\epsilon_2 = \hbar \neq 0$, "Genus zero with insertions", is called Nekrasov-Shatashvili limit and a simple quantum integrable system appears.

3.) The Alday, Gaiotto and Tachikawa conjecture identifies

$$\epsilon_1 = g_s\sqrt{\beta} \quad \epsilon_2 = \frac{g_s}{\sqrt{\beta}}\,,$$

where the Liouville parameters are $c = 1 + 6Q^2$ and $Q = \beta^{\frac{1}{2}} + \beta^{-\frac{1}{2}}$.

Interpretion of (ϵ_1, ϵ_2):

1.) Localization param. in Nekrasovs instanton calc: $\tilde{\mathcal{M}}_k$ moduli space of instanton ($F_A^+ = 0$) in $SU(N)$ bundle over $S^4 = \mathbb{R}^4 \cap \infty$, with $k = -\frac{1}{8N\pi^2}\int_{\mathbb{R}^4} F_A \wedge F_4 \in \mathbb{N}$
Gauge group and spacetime symmetries act on $\tilde{\mathcal{M}}_k$

$$SU(N) \times \mathbb{T}(SO(4)) : \tilde{\mathcal{M}}_k \to \tilde{\mathcal{M}}_k,$$

here $\mathbb{T}(SO(4)) = U(1)_+ \times U(1)_-$ is parametrized by (ϵ_1, ϵ_2) via

$$\epsilon_\pm = \frac{1}{2}(\epsilon_1 \pm \epsilon_2)$$

$$Z(a,\epsilon_1,\epsilon_2,q) = \sum_{k=0}^{\infty} q^k \int_{\tilde{\mathcal{M}}_k} ev(V \otimes S)$$

can be calculated by equivariant localization in terms of partions Y_i, $i = 1, \ldots, N$. E.g. SU(2) with massive m_k flavors in the fundamental

$$Z(a,m,\epsilon,q) = \sum_{Y_1,Y_2} \lambda^{|Y_1|+|Y_2|} f^{Y_1,Y_2}(a,m,\epsilon) f^{Y_2,Y_1}(-a,m,\epsilon)\,,$$

$\lambda = q = e^{2\pi i \tau_{uv}}$ is the UV coupling for the conformal cases ($N_f = 2N$) and the mass scale for the non-conformal cases ($N_f < 2N$). $S = \mathbb{C}^{N_f}$ is the flavor space.

Note that in the Nekrasov partition function $(\epsilon_1, \epsilon_2) \to (-\epsilon_1, -\epsilon_2)$ is not a symmetry, i.e. $F^{(\frac{i}{2},j)} \neq 0$ for i odd.

However a choice of embedding the $U(1)_+$ in the R-symmetry group changes the identification of the flavor masses

$$m'_k = m_k + \rho(\epsilon_1 + \epsilon_2)$$

For $\rho = 1$ the symmetry is restored Göttsche, Nakajima, Yoshioka 2010. This also the physical mass definition of AGT.

Refined M2 brane BPS states Gopakumar, Vafa, ...

M-theory compactification on

$$S^1 \times (\mathrm{CY3-fold}) \times (\mathrm{Taub-Nut}_4)$$

$U(1)_1 \times U(1)_2$ action on TN_4 (e.g. $\mathbb{R}^4 = \mathbb{C}^2$)
$(z_1, z_2) \to (e^{i\epsilon_1} z_1, e^{i\epsilon_2} z_2)$.

Supersymmetry requires $\epsilon_1 = -\epsilon_2$ unless one has an $U(1)_T$-symmetry acting on the CY3-fold $\to$ BPS counting only sensible on non-compact CY.

BPS states have

- Charge $\beta \in H_2(M, \mathbb{Z})$,
- Spin in SO(4)=SU(2)$_+\times$SU(2)$_-$,
- Mass=$\int_\beta \omega$, or $\int_{\Gamma_3} \Omega$ on the mirror
- Multiplicity of charge and spin reps. $n^\beta_{j_+,j_-} \in \mathbb{Z}$.

The latter is counted by BPS index ($q_\pm := e^{-\epsilon_\pm}$):

$$Z(a,\epsilon) = \mathrm{Tr}_{\mathcal{H}}(-1)^{2(m_+ + m_-)} q_+^{2m_+} q_-^{2m_-} e^{-a\cdot\beta},$$

where $m_\pm$ denotes the highest spin $j^3_\pm$ in a given spin multiplet.

Using a Schwinger loop calculation Gopakumar, Hollowood,Iqbal and Vafa 99/03 this can be expressed as

$$\log(Z) = \sum_{\beta,k=1,j_\pm=0}^{\infty} \prod_{\pm} \left(\sum_{m_\pm=-j_\pm}^{j_\pm} q_\pm^{2km_\pm} \right) \frac{n^\beta_{j+,j_-} e^{-ka\cdot\beta}}{4k \sinh(\frac{k\epsilon_1}{2}) \sinh(\frac{k\epsilon_2}{2})}$$

This expansion in manifestly symmetric under $(\epsilon_1, \epsilon_2) \to (-\epsilon_1, -\epsilon_2)$. To make contact with the gauge theory the physical mass has to be used Hollowood,Iqbal and Vafa 2003, Krefl and Walcher:arXiv:1010.2635 .

Behaviour at the conifold points:

Schwinger loop result with a single light hypermultiplet with mass $a_D = \int_{S^3} \Omega \to 0$ for CY 3-fold (resp. $a_D = \int_{S^1} \lambda$),

$$\begin{aligned} F(s,\lambda,a_D) &= -\int_0^\infty \frac{ds}{s} \frac{\exp(-s a_D)}{4\sin(s\epsilon_1/2)\sin(s\epsilon_2/2)} + \mathcal{O}(a_D^0) \\ &= \Big[-\frac{1}{12} + \frac{1}{24} s\lambda^{-2}\Big] \log(a_D) \\ &\quad + \Big[-\frac{1}{240}\lambda^2 + \frac{7}{1440} s - \frac{7}{5760} s^2 \lambda^{-2}\Big] \frac{1}{a_D^2} \\ &\quad + \text{contributions to } 2(g+n)-2 > 2\,. \end{aligned}$$

The leading behaviour $F^{(n,g)} = \frac{N^{(n,g)}}{a_D^{2(g+n)-2}} + \mathcal{O}(a_D^0)$, i.e. the absence of subleading poles is the called gap of $F^{(n,g)}$ at $a_D \to 0$. It will be crucial to solve the models.

The generalized holomorphic anomaly equation

B-model definition of the $F^g(a) = F^{(0,g)}(a)$

$$F^g(a) = \int_{\overline{\mathcal{M}}_g} \left\langle \prod_{k=1}^{3g-3} \beta^k \bar{\beta}^k \right\rangle_g \cdot [dm \wedge d\bar{m}] \,,$$

The contraction of the coodinates $m_k, \bar{m}_k$ with the genus g worldsheet correlator of $\beta^k = \int_{\Sigma_g} G^- \mu^k$, $\bar{\beta}^k = \int_{\Sigma_g} \bar{G}^- \bar{\mu}^k$ with gives a real $6g-6$ form on the compactified moduli space $\overline{\mathcal{M}}_g$ of the g Riemann surface Σ_g.

An infinitessimal anholomorphic perturbation

$S(t_i, \bar{t}_i) = S(t_i) + \bar{t}^i \int_{\Sigma_g} \bar{\mathcal{O}}_i^{(2)}$, with

$$\overline{\mathcal{O}_{\bar{\imath}}^{(2)}} = \{G_0^+, [\bar{G}_0^+, \bar{\mathcal{O}}_{\bar{\imath}}^{(0)}]\} \mathrm{dzd\bar{z}} \ ,$$

correspond to an insertions of exact forms. The deformation receives contributions form the boundaries. This leads to the Holomorphic Anomaly Equation

Bershadski, Cecotti, Ooguri, Vafa 93

$$\bar{\partial}_{\bar{\imath}} F^g = \frac{1}{2} \bar{C}_{\bar{\imath}}^{jk} \big(D_j D_k F^{g-1} + \sum_{h=0}^{g-1} D_j F^h D_k F^{g-h} \big) \, , \quad g > 1 \, .$$

Defining for $g \geq 1$

$$F^{(n,g)}(t) = \int_{\overline{\mathcal{M}_g}} \left\langle \mathcal{O}^n \prod_{k=1}^{3g-3} \beta^k \bar{\beta}^k \right\rangle_g \cdot [dm \wedge d\bar{m}] \,,$$

and for $g = 0$

$$F^{(n+1,0)} = \langle \phi^{(0)}(0)\phi^{(0)}(1)\phi^{(0)}(\infty)\mathcal{O}^n \rangle_{g=0} \ .$$

where the field operator $\mathcal{O}$ should come from integrating a 2-form field over the Riemann surface $\mathcal{O} = \int_{\Sigma_g} \phi^{(2)}$, and $\phi^{(2)}$ emerges as usual from the descend equation from $\phi^{(0)}$. If the latter has no contact term with the fields in the chiral ring on gets Minxin and AK: arXiv:1009.1126

$$\bar{\partial}_{\bar{i}} F^{(n,g)} = \frac{1}{2} \bar{C}_{\bar{i}}^{jk} \Big(D_j D_k F^{(n,g-1)} + {\sum_{m,h}}' D_j F^{(m,h)} D_k F^{(n-m,g-h)} \Big) \,,$$

where the prime denotes omission of $(m,h) = (0,0)$ and $(m,h) = (n,g)$ in the sum. The first term on the right hand side is set to zero if $g = 0$. This is supplemented by the anomaly for $g = 1$

$$\partial_i \bar{\partial}_{\bar{j}} F^1 = \frac{1}{2} C_{ijk} C_{\bar{j}}^{jk} - \frac{\chi}{24} G_{i\bar{j}} \,.$$

A different generalized holomorphic equation was proposed in Krefl and Walcher arXiv:1007.0263. It breaks the $(\epsilon_1, \epsilon_2) \to (-\epsilon_1, -\epsilon_2)$ symmetry and leads for $N_f = 1$ to the Nekrasov partition function with the non-physical masses, which has no obvious modular properties.

The $F^{(n,0)}(a)$ amplitudes fullfill an

$$\bar{\partial}_{\bar{i}} F^{(n,0)} = \frac{1}{2} \bar{C}_{\bar{i}}^{jk} \sum_{m=1}^{n-1} D_j F^{(m,0)} D_k F^{(n-m,0)}$$

anomaly equation that was observed for counting higher rank sheafs on surfaces.

Direct integration of the holomorphic anomaly equations

The direct integration formalism described below applies to theories defined by a Riemann surface $\mathcal{C}_{g'>0}$ and a meromorphic differential λ

- $N = 2$ gauge theories: $\mathcal{C}_{g'}$ Seiberg-Witten curve. λ Seiberg-Witten differential.
- Matrix models: $\mathcal{C}_{g'}$ Spectral curve. λ defines filling factions.
- Non-compact CY3fold: $\mathcal{C}_{g'}$: H(x,y)=0 where $uv +$

$H(x, y) = 0$ is the mirror manifold. $\lambda = \frac{\mathrm{dx}}{y}$

The goal is to derive Z entirely from the curve. For simplicity we will assume $g = 1$. Then we can bring all $\mathcal{C}$ into Weierstrassform

$$y^2 = 4x^3 - g_2(u)x - g_3(u)$$

The classical geometry WS $g = 0$ sector is determined by the J-function which has simple pole at the discriminant

Δ

$$J(\tau) = \frac{E_4(\tau)^3}{E_4(\tau)^3 - E_6(\tau)^2} = \frac{g_2(u)^3}{g_2(u)^3 - 27 g_3(u)^2}\,.$$

where $E_4(\tau)$ and $E_6(\tau)$ are the Eisenstein series of weight 4 and 6, respectively, and the holomorphic period $(\partial a/\partial u) = \int_A \mathrm{dx}/\mathrm{y}$ is given by

$$\frac{da}{du} = c_1 \sqrt{\frac{g_2(u)}{g_3(u)} \frac{E_6(\tau)}{E_4(\tau)}}\,.$$

This determines the prepotential

$$\tau = -\frac{c_0}{2\pi i}\frac{\partial^2 F^0}{\partial a^2},$$

metric on moduli space via

$$G_{a\bar{a}} = 2\partial_a\partial_{\bar{a}}\mathrm{Re}(\bar{a}\partial_a F^0) = \frac{4\pi}{c_0}\tau_2\,.$$

and the three point function

$$C_{aaa} = \frac{\partial^3 F^0}{\partial a^3} = -\frac{2\pi i}{c_0}\frac{d\tau}{da}\,.$$

The $n+g=1$ sector

$F^{(1,0)}$ has no holomorphic anomaly and is determined by the boundary conditions. The Schwinger-loop calculation yields $F^{(1,0)}=\frac{1}{24}\log(a_D)$ and regularity at other singularity dictates

$$F^{(1,0)}=\frac{1}{24}\log(\Delta)f(q)$$

$F^{(0,1)}$ is obtained by integrating the $g=1$ hol. anomaly and obeying the conifold behaviour $F^{(0,1)}=-\frac{1}{12}\log(a_d)$

yielding

$$F^{(0,1)} = -\frac{1}{2}\log(G_{u\bar{u}}|\Delta|^{\frac{1}{3}})\,,$$

so that one gets in the holomorphic limit

$$\lim_{\tau_2\to\infty} F^{(0,1)} = -\frac{1}{2}\log\left(\frac{da}{du}\right) - \frac{1}{12}\log(\Delta)\ .$$

The $n+g>1$ sector The covariant derivative in the τ variable is proportional to the so called Mass derivative

$$\frac{1}{2\pi i}D_\tau = \hat{D}_\tau = \hat{\partial}_\tau - \frac{k}{4\pi\tau_2}\,.$$

The propagator S^{aa}, which can be used to solve the holomorphic anomaly e.q., is proportional to the almost holomorphic form

$$\hat{E}_2(\tau,\bar{\tau}) = E_2(\tau) - \frac{3}{\pi\tau_2}\,.$$

The Mass derivatives closes on the ring of almost modular forms whose only non-holomorphic generator is the propagator, so that

$$\frac{\partial}{\partial\bar{\tau}} = \frac{3i}{2\pi\tau_2^2}\frac{\partial}{\partial\hat{E}_2}\,.$$

This allows to write the holomorphic anomaly as

$$24\frac{\partial F^{(n,g)}}{\partial \hat{E}_2} = c_0\left(\frac{\partial^2 F^{(n,g-1)}}{\partial a^2} + {\sum_{m,h}}' \frac{\partial F^{(m,h)}}{\partial a}\frac{\partial F^{(n-m,g-h)}}{\partial a}\right).$$

Because of the closing of the deriviative on the generators the r.h.s. will always be a polynomial in the generators. It is convenient to write the equation in terms of u, m, introduce

$$X = \frac{g_3(u)}{g_2(u)}\frac{\hat{E}_2(\tau)E_4(\tau)}{E_6(\tau)}\,.$$

to obtain a general

$$F^{(n,g)} = \frac{1}{\Delta^{2(g+n)-2}(u)} \sum_{k=0}^{3g+2n-3} X^k p_k^{(n,g)}(u),$$

where $p_k^{(n,g)}(u)$ are finite polynomials in derivatives of $g_2(u)$ and $g_3(u)$.

(Rigid) Special Geometry implies generally that the covariant derivatives closes on the propagtor

$$D_i S^{kl} = -C_{inm} S^{km} S^{ln} + f_i^{kl}$$

This makes the formulism applicable to $\mathcal{C}_{g'>1}$ and in part to CY3-folds Yamaguchi, Yau, Huang, Quackenbush, AK, Alim, Länge, ...

Fixing of the holomorphic ambiguity: At generic zero u_0 of the descriminant, a single period

$$a_D \sim u - u_0$$

vanishes, i.e. the holomorphic ambiguity can at worse have a pole of order $2(g+n)-2$ at u_0 and we can parameterize it

$$F^{(n,g)}_{hol.amb.} = \frac{1}{\Delta(u)^{2(g+n)-2}} p(u)\,, \qquad (1)$$

with $p(u)$ a holomorphic function of u. Demanding holomophicity and regularity in the limit $u \to \infty$, implies that $p(u)$ is in fact a polynomial in u of degree $(2(g+n)-2)\,d_\Delta - 1$. I.e. the $(2(g+n)-2)\,d_\Delta$ coefficients in$p(u)$ are exactly fixed by the boundary conditions, $(2(g+n)-2)$ via the gap condition at each of the d_Δ zeros.

Applications: Topological Strings on non-compact CY3-folds

Example: $\mathcal{O}(-3) \to \mathbb{P}^2$

$$H(x,y;z) = x + 1 - z\frac{x^3}{y} + y = 0, \qquad \lambda = \log(y)\frac{\mathrm{dx}}{x} \ .$$

Genus 0

$$C_{zzz} = -\frac{1}{3}\frac{1}{z^3(1+27z)}$$

$n + g = 1$:

$$F^{(0,1)} = -\frac{1}{2}\log(\frac{\partial T}{\partial z}) - \frac{1}{12}\log(z^7\Delta),$$

$$F^{(1,0)} = \frac{1}{24}\log(\frac{\Delta}{z}) \ .$$

$n+g=2$

$$\begin{aligned} F^{(0,2)} &= \frac{100X^3-90X^2z^2+30Xz^4+3(9z-1)z^6}{4320z^6\Delta^2} \\ F^{(1,1)} &= \frac{10X^2+5S(108z-1)z^2+2(1-54z)z^4}{1440z^4\Delta^2} \\ F^{(2,0)} &= \frac{10X+(1296z+11)z^2}{11520z^2\Delta^2} \end{aligned}$$

Using the mirror maps at the singular points one can make predictions of refined BPS and refined orbifold BPS invarianst.

E.g. for large radius one gets the results from the refined topological vertex Iqbal, Kozcaz and Vafa: hep-th/0701165 casted in $\Gamma(3)$ modular forms to all order in the modulus.

Other non-compact toric CY3-fold $(\mathcal{O}(-K) \rightarrow \mathbb{F}_n$ have been checked in [**?**].

Applications: N=2 Gauge theories

It is sensible to start with the conformal cases and obtain the asymptotic free cases in limits of gauge coupling and masses (e.g. for $SU(2)$ $N_f = 4$ consider $\lim_{\tau\rightarrow i\infty, m_4\rightarrow\infty} e^{2\pi i\tau} m_4 = \Lambda_3$ etc.)

SU(2) N=4 with adjoint hypermultiplet

$$y^2 = \prod_{i=}^{3} (x - e_i(\tau_{ir})\tilde{u} - \frac{1}{4} e_i(\tau_{ir})\tilde{m}^2),$$

$e_i(\tau_{ir})$, $i = 1, 2, 3$, are the halfperiods

$e_2 - e_1 = \theta_3^4(\tau_{ir})$, $\quad e_3 - e_1 = \theta_2^4(\tau_{ir})$, $\quad e_2 - e_3 = \theta_4^4(\tau_{ir})$.

$e_1 + e_2 + e_3 = 0$ with

$$m = 2\sqrt{2}\tilde{m}(e_2 - e_1), \qquad q(Q) = \frac{e_3 - e_1}{e_2 - e_1} = \frac{\theta_2^4}{\theta_3^4}$$

one gets

$$y^2 = x(x - u - \frac{1}{32}m^2)(x - uq - \frac{1}{32}q^2m^2) \ .$$

Genus zero Eliminating $J(q)$ in favor of τ_{ir}

$$J(\tau_{ir}) = \frac{4m^4(1-q^2+q^4)+8m^2(2-q-q^2+2q^3)u+64(1-q+q^2)u^2)^3}{27(1-q)^2q^2(m^2+8u)^2(m^4q(1+q)+64u^2+8m^2(u+2qu))^2} \ .$$

one gets Nekrasovs partition function $F^{(0,0)}(a,m,Q)$ in$\lambda = Q = e^{2\pi\tau_{ir}}$ after shifting the masses. In this case the theory is truely conformal and the τ_{ir} parameter is identitified with Nekrasovs τ_{uv}.

$n + g = 1$

$$F^{(1,0)} = \frac{1}{24} \log \left(\frac{\Delta}{q^4(1-q)} \right) \; .$$

$$F^{(0,1)} = -\frac{1}{2} \log \left(\frac{da}{du} \right) - \frac{1}{12} \log \left(\frac{\Delta \sqrt{(1-q)}}{q} \right) \; ,$$

In the massless limit

$$F^{(1,0)}(a) = \frac{1}{4} \log(a) + \mathcal{O}(Q), \qquad F^{(0,1)}(a) = 0 + \mathcal{O}(Q) \; .$$

$n + g > 1$

$$
\begin{aligned}
F^{(2,0)} &= \frac{E_2}{24 \cdot 32 a^2}, \\
F^{(1,1)} &= -\frac{E_2}{6 \cdot 32 a^2}, \\
F^{(0,2)} &= 0,
\end{aligned} \tag{2}
$$

$F^{(0,n)} = 0$ is a simple consequence of the holomorphic anomaly equation and the fact that $F^{(0,1)}$ has no a-dependence.

$$
\begin{aligned}
F^{(3,0)} &= -\frac{1}{2^{13}3^2a^4}\left(E_2^2+\frac{13}{5}E_4\right), \\
F^{(2,1)} &= \frac{1}{2^{12}3^2a^4}\left(5E_2^2+\frac{29}{5}E_4\right), \\
F^{(2,1)} &= -\frac{1}{2^{10}3a^4}\left(E_2^2+\frac{1}{5}E_4\right)
\end{aligned} \tag{3}
$$

Note that $E_2 \sim X$ powers grow only with $g+n-1$ in the massless cases. The ambiguity is determined using the mass deformations to achieve generic singularities. In the massive case there are higher powers in E_2 up to $3g+2n-3$, which reproduce in the double scaling pure $SU(2)$ SYM.

SU(2) N=2 with four fundamental hypers

Both conformal theories can be geometrically engineered from the Enriques CY-3-fold, whose fibre over $\mathbb{P}^1$, has the lattice

$$\Gamma_1 = \Gamma^{1,1}_s \oplus \Gamma^{1,1}(2) \oplus E_8(-2)$$

The N=4 gauge bosons come from the $\Gamma^{1,1}(2)$ lattice while the $N_f = 4$ gauge bosons come form $\Gamma^{1,1}_s$. There are two reductions the geometrical reduction in the VM of the first are large radius Kähler parameters, while in the second the volume of the Enriques collapses the VM have to be describes by the mirror geometry.

The massless curves in both case the torus in the base, but in the N=4 the good parameter is $Q = e^{2\pi i\tau_{ir}}$, while one has to use the mirror coordanite $q(Q) = \frac{e_3 - e_1}{e_2 - e_1} = \frac{\theta_2^4}{\theta_3^4}$ in the $N_f = 4$ case.

The mass defomed curve is

$$
\begin{aligned}
y^2 = \;& x(x-u)(x-qu) - x^2(1-q)^2 \sum_{i=1}^{4} \tilde{m}_i^2 \\
& +4x(1-q)q \left(2(1+q) \prod_{i=1}^{4} \tilde{m}_i - (1-q) \sum_{i<j} \tilde{m}_i^2 \tilde{m}_j^2 \right) \\
& -16(1-q)q^2 \left(u \prod_{i=1}^{4} \tilde{m}_i + (1-q) \sum_{i<j<k} \tilde{m}_i^2 \tilde{m}_j^2 \tilde{m}_k^2 \right) ,
\end{aligned}
$$

with $\tilde{m}_i = m_i\sqrt{e_1 - e_2}$. I.e. the IF coupling τ_{ir} is not identified with the UV coupling. The modular properties however are w.r.t. to the IR coupling.

$g + n = 1$

$$F^{(1,0)} = \frac{1}{24}\log\left(\frac{(1-q)^4}{q^2}\Delta\right)$$

and

$$F^{(0,1)} = -\frac{1}{2}\log\left(\frac{\mathrm{d}a}{\mathrm{d}u}\right) - \frac{1}{12}\log\left(\frac{1}{(1-q)^2 q^2}\Delta\right),$$

$g + n > 1$

$$F^{(2,0)} = \frac{E_2}{24 \cdot 32a^2}, \qquad F^{(1,1)} = -\frac{E_2}{6 \cdot 32a^2}, \qquad F^{(0,2)} = 0,$$

Conclusions :

- The generalized holomorphic anomaly equations and the gap conditions determine the partition function of the physical systems, which allow for the (ϵ_1, ϵ_2) refinement:
  - Non-compact CY3-folds
  - N$=$2 gauge theories
  - Matrix models with β-ensemble measures, if the potential is polynomial

  as a global almost holomorphic modular expression over the moduli space.