The global gravitational anomaly of the self-dual field theory

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String-Math 2011, June 8th 2011

MOTIVATION

The self-dual field theory is the theory of an abelian 2ℓ-form gauge field, living on a $4\ell + 2$-dimensional manifold, whose $2\ell + 1$-form field strength obey a self-duality condition: $F = *F$.

Examples:

- The chiral boson in $d = 2$
- The world volume theory of the M-theory and IIA fivebranes
- The RR 4-form gauge field in type IIB supergravity.

The global gravitational anomaly of this theory is still unknown. `Witten'88` made a heuristic proposal, in the case when the middle-degree cohomolgy vanishes. How to get a general formula?

Manifolds of dimension $4\ell + 2$

Consider a compact Riemannian manifold X of real dimension $4\ell + 2$.

Properties specific to this dimension:

- The exterior product gives a symplectic structure ω on $\Omega^{2\ell+1}$.
- The Hodge star operator squares to $-\mathbb{1}$ on $\Omega^{2\ell+1}$, hence defines a complex structure.
- Both structures restrict to $H^{2\ell+1}$

We endow X with a $\mathbb{Z}_2$-valued quadratic refinement q of the intersection product on $H^{2\ell+1}(X,\mathbb{Z})$ modulo 2:

$$q(\rho_1 + \rho_2) - q(\rho_1) - q(\rho_2) = \omega(\rho_1, \rho_2) \mod 2 \ ,$$

$$q_\eta(\rho = \hat{\rho} + \check{\rho}) = \omega(\hat{\rho}, \check{\rho}) + \omega(\eta, \rho) \ , \quad \eta \in H^{2\ell+1}(X,\mathbb{Z}) \ .$$

Riemannian metrics modulo diffeomorphisms

- Let $\mathcal{M}$ be the manifold of Riemannian metrics on X.
- Let $\mathcal{D}^{(2)}$ be the group of diffeomorphisms preserving the quadratic refinements.
- Form the quotient $\mathcal{M}/\mathcal{D}^{(2)}$.
- This quotient space is very singular, because of metrics admitting isometries.
- To get a smooth quotient, consider only diffeomorphisms leaving a point of X and its tangent space fixed.

We would like to discuss four methods to construct line bundles over $\mathcal{M}/\mathcal{D}^{(2)}$.

METHOD 1: PULL-BACK

1st method: Pull back bundles from a finite-dimensional space.

- The action of $D^{(2)}$ on X induces an action on $H^{2\ell+1}$.
- This action is symplectic with respect to ω, preserves the integral cohomology and factors through $\Gamma^{(2)} \subset \mathrm{Sp}(2n, \mathbb{Z})$.
- The Hodge star operator restricts to $H^{2\ell+1}$, and defines a point τ in the Siegel upper-half plane $\mathcal{C}$.

$\Rightarrow$ We have a map $\mathcal{M}/\mathcal{D}^{(2)} \to \mathcal{C}/\Gamma^{(2)}$.

METHOD 1: PULL-BACK

Line bundles over $\mathcal{C}/\Gamma^{(2)}$ have been classified by `Putman'10`, using results of `Sato'08`, at least for $2n := \dim H^{2\ell+1} \geq 6$:

$$0 \to (\mathbb{Z}_2)^{2n^2-n} \times (\mathbb{Z}_4)^{2n} \to \mathrm{Pic}(\mathcal{C}/\Gamma^{(2)}) \stackrel{\mathrm{ch}}{\to} \mathbb{Z} \to 0 \ .$$

- Siegel modular forms can be seen as sections of line bundles over $\mathcal{C}/\Gamma^{(2)}$. The weight of the modular form is $\frac{1}{2}\mathrm{ch}(\mathscr{L})$.
- Let $\mathscr{C}^\eta$ the bundle associated with a theta constant with characteristic η. It generates Pic modulo torsion.
- Let $\mathscr{K}$ the determinant of the Hodge bundle, with fiber $\bigwedge^{2n} H^{2\ell+1}$. $\mathrm{ch}(\mathscr{K}) = 2$.
- $(\mathscr{C}^\eta)^2 \otimes \mathscr{K}^{-1} = \mathscr{F}^\eta$, a flat bundle. $\mathscr{F}^\eta$ is described by a character χ^η of $\Gamma^{(2)}$.

METHOD 2: FAMILIES OF DIRAC OPERATORS

2nd method: Determinant bundles of Dirac operators.

- Construct the fibration

$$(\mathcal{M} \times X)/\mathcal{D}^{(2)} \to \mathcal{M}/\mathcal{D}^{(2)} \; .$$

- See a Dirac operator D on X as a family of Dirac operators parameterized by $\mathcal{M}/\mathcal{D}^{(2)}$.
- This family has a determinant bundle $\mathscr{D}$, which is a line bundle over $\mathcal{M}/\mathcal{D}^{(2)}$.
- $\mathscr{D}$ is endowed with a natural hermitian structure and connection.

Method 2: Families of Dirac operators

Choose for D the Dirac operator coupled to chiral spinors:

$$D : \Omega^{\text{even}}_{\text{SD}} \simeq S^+ \otimes S^+ \to \Omega^{\text{odd}}_{\text{SD}} \simeq S^- \otimes S^+ \ .$$

- Then topologically, $\mathscr{D} \simeq \mathscr{K}^{-1} \otimes \mathscr{D}_+ \simeq \mathscr{K}^{-1}$.
- However their natural connections and hermitian structures do not coincide.

Using work of Bismut-Freed and the Atiyah-Patodi-Singer theorem, the holonomy of the connection along a loop γ in $\mathcal{M}/\mathcal{D}^{(2)}$ reads:

$$\mathrm{hol}_{\mathscr{D}^{-1}}(\gamma) = \exp 2\pi i \frac{1}{4}\left(\sigma_B + 2h + \chi - \int_B L\right)$$

for B a manifold bounding the mapping torus X_γ. L is the Hirzebruch genus of B.

METHOD 3: HOPKINS-SINGER CONSTRUCTION

3rd method: The construction of Hopkins and Singer

`Hopkins-Singer'02` construct a line bundle on the base of any fibration of $4\ell + 2$ manifold equipped with a metric and a refinement of the intersection pairing. Applying it to the same fibration, we get a line bundle $\mathscr{R}^\eta$.

- The holonomies of $\mathscr{R}^\eta$ are given by

$$\mathrm{hol}_{\mathscr{R}^\eta}(\gamma) = \exp 2\pi i \frac{1}{8} \int_B \left(\lambda^2 - L\right)$$

  where λ is a cohomology class defined on B depending on the quadratic refinement.

- $(\mathscr{R}^\eta)^2 \otimes \mathscr{D}$ is a flat bundle with holonomies

$$\mathrm{hol}_{(\mathscr{R}^\eta)^2 \otimes \mathscr{D}}(\gamma) = \exp 2\pi i \frac{1}{4} \left(\int_B \lambda^2 - \sigma_B - 2h - \chi \right)$$

Method 4: Quantum field theory

4th method: Quantum field theory

- The partition function of a quantum field theory defines the section of a line bundle with connection over $\mathcal{M}/\mathcal{D}^{(2)}$.
- If the bundle is not canonically trivialized, there is an *anomaly*.
- The *topological anomaly* is the topological class of the anomaly bundle.
- The *local anomaly* is the curvature of the connection.
- The *global anomaly* is the holonomies of the connection.
- If both the local anomaly and the global anomaly vanish, the anomaly bundle has a canonical trivialization and the partition function can be seen as a function over $\mathcal{M}/\mathcal{D}^{(2)}$.

METHOD 4: QUANTUM FIELD THEORY

The self-dual field theory provides such a bundle $\mathscr{A}^\eta$.

- The local anomaly for a pair of self-dual field coincides with the anomaly of $\mathscr{D}^{-1}$. $\Rightarrow$ modulo torsion, $(\mathscr{A}^\eta)^2 \equiv \mathscr{D}^{-1}$.
- Main result: $\mathscr{A}^\eta \simeq \mathscr{C}^\eta$.
- Using in addition $(\mathscr{C}^\eta)^2 \simeq \mathscr{K} \otimes \mathscr{F}^\eta$, we can deduce that $(\mathscr{A}^\eta)^2 \equiv \mathscr{D}^{-1} \otimes \mathscr{F}^\eta$ and get a formula for the global anomaly of a pair of self-dual fields

$$\mathrm{hol}_{(\mathscr{A}^\eta)^2}(\gamma) = \chi^\eta(\gamma) \exp 2\pi i \frac{1}{4} \left(\sigma_B + 2h + \chi - \int_B L \right)$$

How to take the square root to obtain the global anomaly of a single self-dual field?

Conjectures and holonomy formula

Conjectures (naive?):

- $\mathscr{R}^\eta \simeq \mathscr{A}^\eta (\simeq \mathscr{C}^\eta)$: The Hopkins-Singer bundle coincides topologically with the pull back of the theta bundle.
- $\mathscr{A}^\eta \equiv \mathscr{R}^\eta$, so $\mathscr{F}^\eta \equiv (\mathscr{R}^\eta)^2 \otimes \mathscr{D}$ and

$$\frac{1}{2\pi i} \log \chi^\eta(\gamma) = \frac{1}{4} \left(\int_B \lambda^2 - \sigma_B - 2h - \chi \right)$$

If true, the Hopkins-Singer formalism provides a formula for the global gravitational anomaly of the self-dual field theory:

$$\mathrm{hol}_{\mathscr{A}^\eta}(\gamma) = \exp 2\pi i \frac{1}{8} \int_B \left(\lambda^2 - L \right)$$

CONCLUSION

Summary of the results:

- We determined the topological anomaly of the self-dual field theory.
- We have a conjectural formula for the global anomaly.