Higher rank stable pairs and virtual localization

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Definition 1: Let X be a nonsingular projective Calabi-Yau 3-fold over $\mathbb{C}$ (i.e $K_X \cong \mathcal{O}_X$ and $\pi_1(X) = 0$ which implies $\mathrm{H}^1(\mathcal{O}_X) = 0$) with a fixed polarization L . A holomorphic triple supported over X is given by (E_1, E_2, ϕ) consisting of a torsion free coherent sheaf E_1 and a pure sheaf with one dimensional support E_2, together with a holomorphic morphism $\phi : E_1 \to E_2$. A homomorphism of triples from $(\acute{E}_1, \acute{E}_2, \acute{\phi})$ to (E_1, E_2, ϕ) is a commutative diagram:

$$\begin{array}{ccc} \acute{E}_1 & \xrightarrow{\acute{\phi}} & \acute{E}_2 \\ \downarrow & & \downarrow \\ E_1 & \xrightarrow{\phi} & E_2 \end{array}$$

Definition 2: A frozen-triple of class β and of fixed Hilbert polynomial P_2 is a frozen-triple (E_1, E_2, ϕ) such that the Hilbert polynomial of E_2 is equal to P_2 and $\beta = \mathrm{ch}_2(E_2)$. Having fixed r in $E_1 \cong \mathcal{O}_X^{\oplus r}(-n)$, we denote these frozen triples as frozen triples of type (P_2, r).

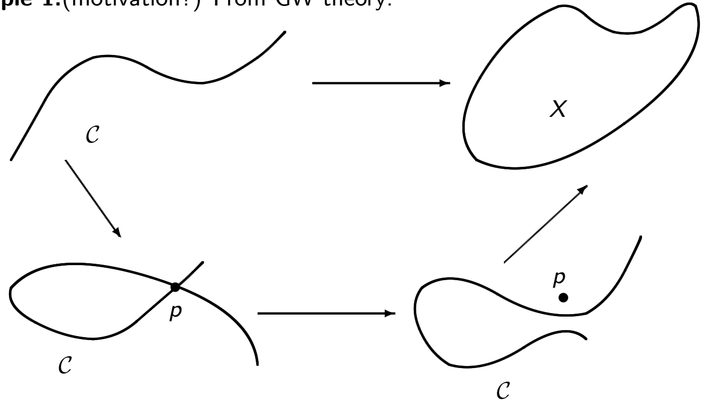
Definition 3: A *highly frozen triple* is a quadruple (E_1, E_2, ϕ, ψ) where (E_1, E_2, ϕ) is a frozen triple as in Definition 2 and $\psi : E_1 \xrightarrow{\cong} \mathcal{O}_X(-n)^{\oplus r}$ is a fixed choice of isomorphism (a choice of trivialization of E_1). A morphism between highly frozen triples $(E_1', E_2', \phi', \psi')$ and (E_1, E_2, ϕ, ψ) is a morphism $E_2' \xrightarrow{\rho} E_2$ such that the following diagram is commutative.

$$\begin{array}{ccccc} \mathcal{O}_X(-n)^{\oplus r} & \xrightarrow{\psi'^{-1}} & E_1' & \xrightarrow{\acute{\phi}} & E_2' \\ id \downarrow & & \downarrow & & \downarrow \rho \\ \mathcal{O}_X(-n)^{\oplus r} & \xrightarrow{\psi^{-1}} & E_1 & \xrightarrow{\phi} & E_2 \end{array}$$

A frozen triple of rank 1 is given as an n-twisted Pandharipande-Thomas stable pair

$$\mathcal{O}_X(-n) \to E_2.$$

Example 1:(motivation?) From GW theory:



$$0 \to \mathcal{O}_\mathcal{C} \to \mathcal{O}_\mathcal{C}(p) \to \mathcal{O}_p \to 0$$

Viewed as exact sequence of sheaves of $\mathcal{O}_X$-modules we get:

$$0 \to \mathcal{I}_{\mathcal{C}} \to [\mathcal{O}_X \to i_*\mathcal{O}_{\mathcal{C}}(p)] \to i_*\mathcal{O}_p \to 0.$$

The corresponding moduli spaces of FT and HFT

1. Let $\mathfrak{M}^{(P_2,r,n)}_{s,HFT}(\tau')$ denote moduli stack of highly frozen triples with given data (of type) (P_2, r).
2. Let $\mathfrak{M}^{(P_2,r,n)}_{s,FT}(\tau')$ denote moduli stack of highly frozen triples with given data (of type) (P_2, r).

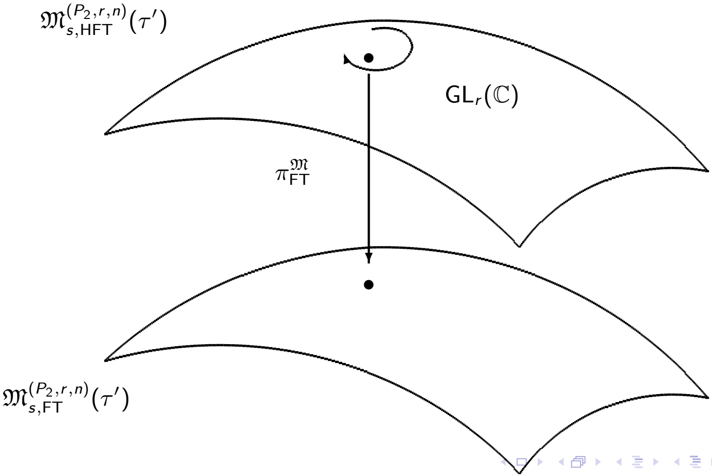
We prove the following theorems about the stacky structure of these moduli stacks:

Theorem 1: There exists a natural diagram:

$$\begin{array}{ccc} \mathfrak{M}^{(P_2,r,n)}_{s,\mathrm{HFT}}(\tau') & \xrightarrow{\quad g \quad} & pt = \mathrm{Spec}(\mathbb{C}) \\ \downarrow \pi^{\mathfrak{M}}_{\mathrm{FT}} & & \downarrow i \\ \mathfrak{M}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau') & \xrightarrow{\quad \acute{g} \quad} & \mathcal{B}\,\mathrm{GL}_r(\mathbb{C}) = \left[\frac{\mathrm{Spec}(\mathbb{C})}{\mathrm{GL}_r(\mathbb{C})}\right] \end{array}, \tag{1}$$

which is a fibered diagram in the category of stacks. In particular $\mathfrak{M}^{(P_2,r,n)}_{s,\mathrm{HFT}}(\tau')$ is a $\mathrm{GL}_r(\mathbb{C})$-torsor over $\mathfrak{M}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau')$. It is true that locally in the flat topology $\mathfrak{M}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau') \cong \mathfrak{M}^{(P_2,r,n)}_{s,\mathrm{HFT}}(\tau') \times \left[\frac{\mathrm{Spec}(\mathbb{C})}{\mathrm{GL}_r(\mathbb{C})}\right]$. This isomorphism does not hold true globally unless $r = 1$.

Theorem 2: The moduli stacks $\mathfrak{M}^{(P_2,r,n)}_{s,HFT}(\tau')$ and $\mathfrak{M}^{(P_2,r,n)}_{s,FT}(\tau')$ are given as algebraic quotient stacks. Moreover $\mathfrak{M}^{(P_2,r,n)}_{s,HFT}(\tau')$ is a DM stack while $\mathfrak{M}^{(P_2,r,n)}_{s,HFT}(\tau')$ has stacky structure of an Artin stack.



Definition 4: Following G. Laumon-L. Moret-Bailly and Olsson by definition a perfect deformation-obstruction theory for an Artin stack (in our case $\mathfrak{M}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau'))$ is given by a perfect 3-term complex $\mathbb{E}^{\bullet}$ of amplitude $[-1,1]$ and a map in the derived category:

$$\mathbb{E}^{\bullet} \xrightarrow{\phi} \mathbb{L}^{\bullet}_{\mathfrak{M}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau')}$$

such that $h^1(\phi)$ and $h^0(\phi)$ are isomorphisms and $h^{-1}(\phi)$ is an epimorphism.

Here $\mathbb{L}^{\bullet}_{\mathfrak{M}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau')}$ is the truncated cotangent complex of the Artin moduli stack of τ'-stable frozen triples concentrated in degrees -1, 0 and 1 which has the form:

$$\mathbb{L}^{\bullet}_{\mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau')} : \mathcal{I}/\mathcal{I}^2 \to \Omega_{\mathfrak{A}} \mid_{\mathfrak{H}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau')} \to \mathfrak{gl}_r(\mathbb{C})^{\vee} \otimes \mathcal{O}_{\mathfrak{M}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau')},$$

By Theorem 2 $\mathfrak{M}^{(P_2,r,n)}_{s,\mathrm{HFT}}(\tau')$ is a DM stack.
In this situation the truncated cotangent complex takes the form:

$$\mathbb{L}^{\bullet}_{\mathfrak{M}^{(P_2,r,n)}_{s,\mathrm{HFT}}(\tau')} : \mathcal{I}/\mathcal{I}^2 \longrightarrow \Omega_{\mathfrak{A}} \mid_{\mathfrak{M}^{(P_2,r,n)}_{s,\mathrm{HFT}}(\tau')} .$$

Here following Behrend and Fantechi, a perfect deformation-obstruction theory is given by a perfect 2 term complex $\mathbb{G}^{\bullet}$ and a map in the derived category:

$$\mathbb{G}^{\bullet} \xrightarrow{\phi} \mathbb{L}^{\bullet}_{\mathfrak{M}^{(P_2,r,n)}_{s,\mathrm{HFT}}(\tau')},$$

such that $h^0(\phi)$ is an isomorphism and $h^{-1}(\phi)$ is an epimorphism.

Our goal is to construct a suitable complex such as $\mathbb{G}^\bullet$ or $\mathbb{E}^\bullet$ for HFT or FT. We obtain these complexes by deforming universal objects over $\mathfrak{M}^{(P_2,r,n)}_{s,\mathrm{HFT}}(\tau')$ and $\mathfrak{M}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau')$.

Let $\mathbb{I}^\bullet : \mathcal{O}^{\oplus r}_{X\times\mathfrak{M}^{(P_2,r,n)}_{s,\mathrm{HFT}}(\tau')}(-n) \to \mathbb{F}$ denote the universal highly frozen triple. By deforming this complex we obtain a complex whose cohomologies over each point $I^\bullet : \mathcal{O}_X^{\oplus r}(-n) \to F$ in the moduli stack give:

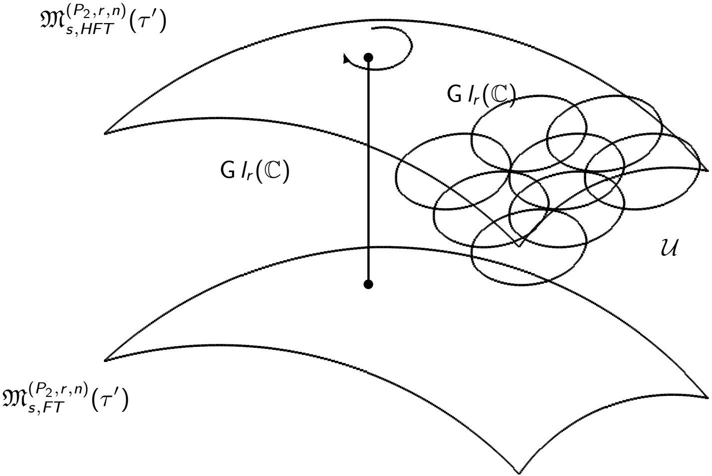
1. Deformations (Given by): $\mathrm{Hom}(I^\bullet, F)$
2. Obstructions (Given by): $\mathrm{Ext}^1(I^\bullet, F)$
3. Higher obstructions (Given by): $\mathrm{Ext}^2(I^\bullet, F)$

For FT deformations are given by: $\mathrm{Hom}(I^\bullet, F)/\mathfrak{gl}_r(\mathbb{C})$. This comes from Theorem 2 which states that $\mathfrak{M}^{(P_2,r,n)}_{s,\mathrm{HFT}}(\tau')$ is a $\mathrm{GL}_r(\mathbb{C})$-torsor over $\mathfrak{M}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau')$.

We switch perspective and think of frozen and highly frozen triples as objects in derived category. By deforming the universal objects (in derived category) over $\mathfrak{M}^{(P_2,r,n)}_{s,\mathrm{FT}}(\tau')$ or $\mathfrak{M}^{(P_2,r,n)}_{s,\mathrm{HFT}}(\tau')$ we obtain a complex whose cohomologies over a point in each moduli stack are given by 4 terms:

1. $\mathrm{Ext}^0(I^\bullet, I^\bullet)_0$.
2. $\mathrm{Ext}^1(I^\bullet, I^\bullet)_0$.
3. $\mathrm{Ext}^2(I^\bullet, I^\bullet)_0$
4. $\mathrm{Ext}^3(I^\bullet, I^\bullet)_0$

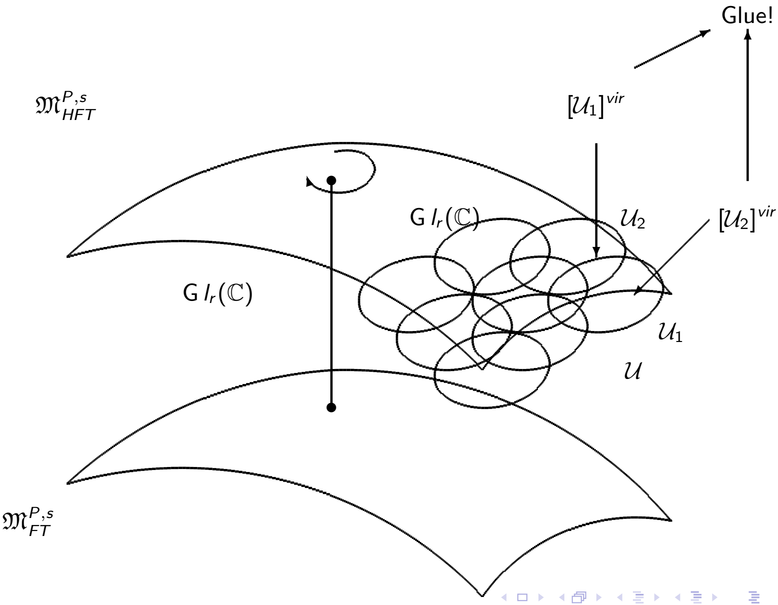
We can show that in our setup $\mathrm{Ext}^1(I^\bullet, I^\bullet)_0 \cong \mathrm{Hom}(I^\bullet, F)/\,\mathrm{gl}_r(\mathbb{C})$. Hence we can not use objects in derived category for HFT but we can use them for FT. We describe our strategy as follows:


$\mathfrak{M}^{(P_2,r,n)}_{s,HFT}(\tau')$
$Gl_r(\mathbb{C})$
$Gl_r(\mathbb{C})$
$\mathcal{U}$
$\mathfrak{M}^{(P_2,r,n)}_{s,FT}(\tau')$

Locally over $\mathcal{U}$ we construct a suitable truncated deformation-obstruction theory by pulling back $\mathbb{E}^{\bullet}$ via $\pi^{\mathfrak{M}}_{FT}:\mathfrak{M}^{(P_2,r,n)}_{s,HFT}(\tau')\longrightarrow\mathfrak{M}^{(P_2,r,n)}_{s,FT}(\tau')$ and the local truncation of $(\pi^{\mathfrak{M}}_{FT})^{*}\mathbb{E}^{\bullet}$.

Theorem 3: Consider the 4-term deformation obstruction theory $\mathbb{E}^{\bullet\vee}$ of perfect amplitude $[-2,1]$ over $\mathfrak{M}^{(P_2,r,n)}_{s,\text{FT}}(\tau')$.

1. Locally in the étale topology over $\mathfrak{H}^{(P_2,r,n)}_{s,\text{HFT}}(\tau')$ there exists a perfect two-term deformation obstruction theory of perfect amplitude $[-1,0]$ which is obtained from the suitable local truncation of the pullback $(\pi^{\mathfrak{M}}_{\text{FT}})^{*}\mathbb{E}^{\bullet\vee}$.
2. This local theory defines a globally well-behaved virtual fundamental class over $\mathfrak{M}^{(P_2,r,n)}_{s,\text{HFT}}(\tau')$.


Glue!
$[\mathcal{U}_1]^{vir}$
$\mathfrak{M}^{P,s}_{HFT}$
$Gl_r(\mathbb{C})$
$Gl_r(\mathbb{C})$
$\mathcal{U}_2$
$[\mathcal{U}_2]^{vir}$
$\mathcal{U}_1$
$\mathcal{U}$
$\mathfrak{M}^{P,s}_{FT}$

Virtual localization:

The torus fixed locus of the moduli stack of highly frozen triples consists of those torus-equivariant HFT's which are written as direct sum of PT stable pairs, i,e:

$$[\mathcal{O}_X(-n)^{\oplus r} \to F]^{\mathbf{T}} = \bigoplus_{i=1}^{r} [\mathcal{O}_X(-n) \to F_i]$$

Virtual localization for rank=4:(i,e $\mathcal{O}_X(-n)^{\oplus 4} \to F$)

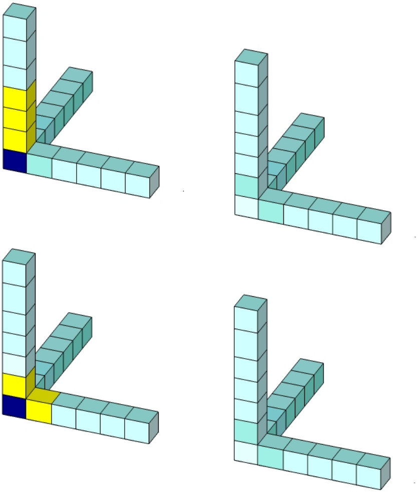


Figure: Allowable configurations for $l = 4$, Cases 1 and 2

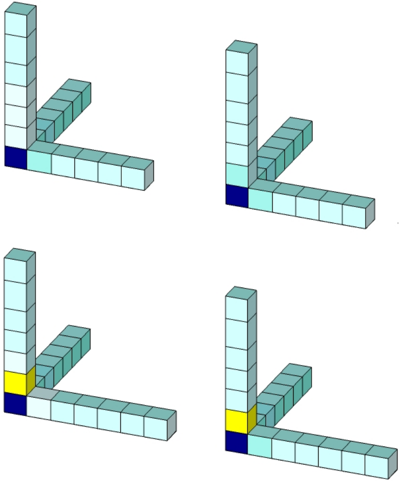


Figure: Allowable configurations for $l = 4$, Cases 3 and 4