

On 2d TQFT whose values are hyperkähler cones

Yuji Tachikawa (IAS & IPMU)

at String-Math 2011, U. Penn

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On 2d TQFT whose values are holomorphic symplectic varieties

Yuji Tachikawa (IAS & IPMU)

in collaboration with **Greg Moore** (Rutgers)

at String-Math 2011, U. Penn

- I'll give a *mathematical reformulation* of the *known results in string theory* literature.

*Argyres-Seiberg, Argyres-Wittig,
Gaiotto-Witten,
Gaiotto-Neitzke-YT,
Benvenuti-Benini-YT,
Chacaltana-Distler, ...*

- I'll start in a *stringy* language, and gradually make it *mathematically more precise*.

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Wrap a 6d $N=(2,0)$ theory on a Riemann surface C with punctures.

This gives a 4d $N=2$ theory depending on the complex structure of C .

Then, the 4d $N=2$ theory decomposes nicely into two, when the surface C is pinched into two surfaces.

- Mathematically, there should be a functor \mathcal{F}_G from the *category of Riemann surfaces* to the *category of 4d $N=2$ theories* for $G=A_n, D_n, E_n$

$$\begin{aligned}
 & \mathcal{F}_G \left[\text{Two circles with dots, joined at a neck} \right] \\
 & \quad \text{Gluing parameter } q \\
 & = \mathcal{F}_G \left[\text{Circle with dots and a pink dot} \right] \times \mathcal{F}_G \left[\text{Circle with dots and a pink dot} \right] \\
 & \quad \text{gauged with the coupling constant } q
 \end{aligned}$$

The diagram illustrates the mapping of a genus-2 surface (two circles joined at a neck) to a product of two gauged punctured surfaces. The genus-2 surface is shown with two circles, each containing two black dots. A double-headed arrow below the neck indicates the gluing parameter q . This is equated to the product of two surfaces, each represented as a circle with three dots (two black and one pink). A dashed line connects the pink dots of the two surfaces, with the label 'gauged with the coupling constant q ' below it.

- The problem is that the *category of 4d N=2 theories* is yet to be defined.
- However, some part of it can still be formulated rigorously.

Take a Riemannian manifold M

choose an Abelian
group A

$H^*(M, A)$
: A -module

$\dim(M)$
: +ve integer

$\text{vol}(M)$
: real number

Various objects associated functorially,
depending on various amount of structures on M

Take a 4d N=2 theory T

choose a 4-mfd X

$Z(T, X)$

: complex number

$\text{Coulomb}(T)$

: special Kähler mfd

$\text{Higgs}(T)$

: hyperkähler mfd

Various objects associated functorially,
depending on various amount of structures on T

- Let us compose the Gaiotto functor \mathcal{F}_G from the *category of Riemann surfaces* to the *category of 4d $N=2$ theories* with the functor $Z(\cdot, S^4)$.

$$\begin{aligned}
 & Z(\mathcal{F}_G[\text{Diagram}], S^4) \\
 & \quad \text{Gluing parameter } q \\
 & = \int_{-\infty}^{\infty} q^{-x^2} Z(\mathcal{F}_G[\text{Diagram 1}], S^4) Z(\mathcal{F}_G[\text{Diagram 2}], S^4) dx
 \end{aligned}$$

The diagram in the first line consists of two circles joined at a point, with two black dots in each circle. A double-headed arrow below it is labeled "Gluing parameter q ".

The first diagram in the second line is a single circle with three black dots and one pink dot labeled x .

The second diagram in the second line is a single circle with four black dots and one pink dot labeled x .

- The functor $Z_{\mathcal{F}_G}$ is
 from the *category of Riemann surfaces*
 to the *category of vector spaces*.
 i.e. this is a *2d conformal field theory*.

$$\begin{aligned}
 & Z(\mathcal{F}_G [\text{Diagram}], S^4) \\
 & \quad \text{Gluing parameter } q \\
 & = \int_{-\infty}^{\infty} q^{-x^2} Z(\mathcal{F}_G [\text{Diagram 1}], S^4) Z(\mathcal{F}_G [\text{Diagram 2}], S^4) dx
 \end{aligned}$$

The diagram in the first line shows two circles connected by a neck, with two black dots in each circle. A double-headed arrow below it is labeled "Gluing parameter q ".

The first diagram in the second line shows a single circle with three black dots and a pink dot labeled x .

The second diagram in the second line shows a single circle with four black dots and a pink dot labeled x .

- This is the essence of the *AG* correspondence*.

- Let us compose the Gaiotto functor \mathcal{F}_G from the *category of Riemann surfaces* to the *category of 4d $N=2$ theories* with the functor $\text{Higgs}(\cdot)$ to the category of hyperkähler manifolds.

$$\begin{aligned}
 & \text{Higgs}(\mathcal{F}_G [\text{Diagram}]) \\
 & \quad \text{Gluing parameter } q \\
 & = \text{Higgs}(\mathcal{F}_G [\text{Diagram 1}]) \times \text{Higgs}(\mathcal{F}_G [\text{Diagram 2}]) \quad // \quad G
 \end{aligned}$$

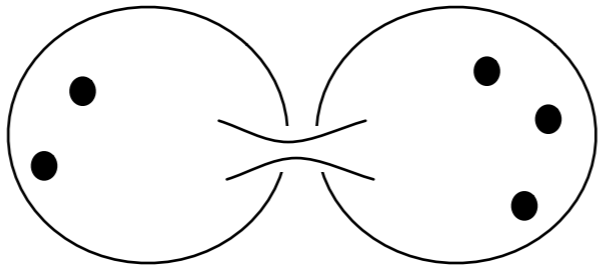
The diagram in the first line shows two circles connected by a neck, with two black dots in each circle. A double-headed arrow below the neck is labeled "Gluing parameter q ".

The first diagram in the second line shows a single circle with three black dots (two on the left, one on the right) and one pink dot on the right.

The second diagram in the second line shows a single circle with one pink dot on the left and three black dots on the right.

- This is independent of q as holomorphic symplectic varieties.
- $\eta_G = \text{Higgs}_{\mathcal{F}_G}$ is a functor from the *category of 2-cobordisms* to the *category of hol. sympl. varieties*.

$$\text{Higgs}(\mathcal{F}_G [\text{Diagram}])$$



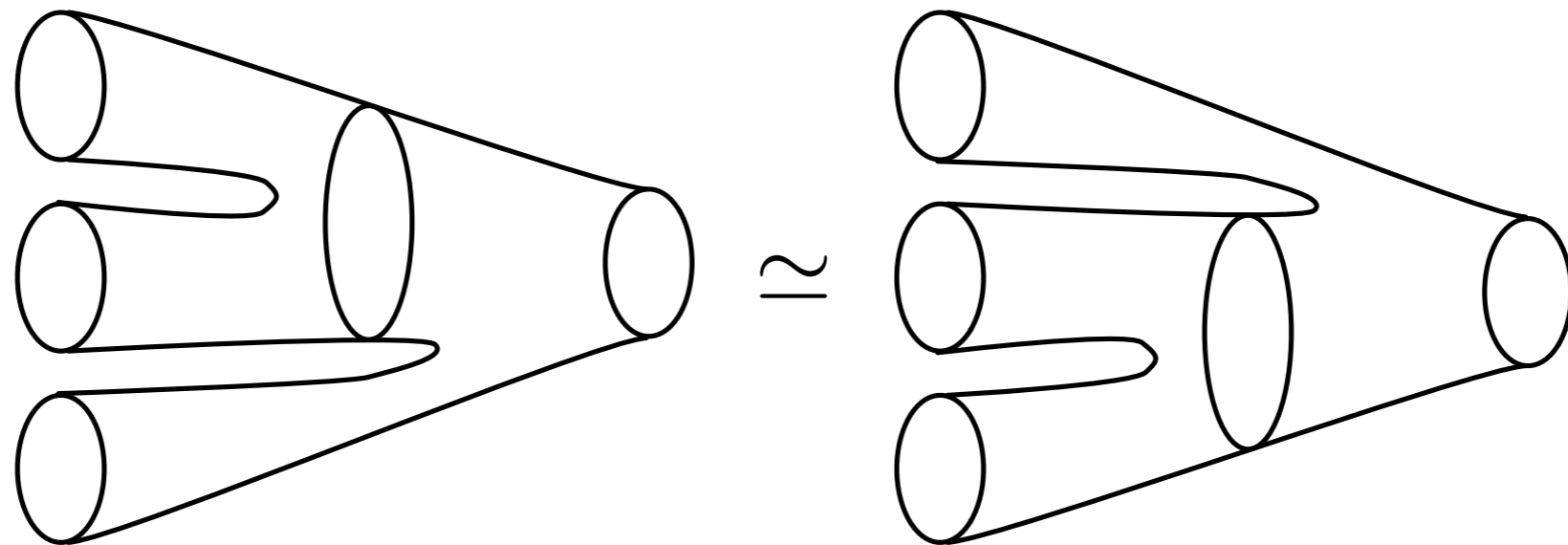
 Gluing parameter q

$$= \text{Higgs}(\mathcal{F}_G [\text{Diagram 1}]) \times \text{Higgs}(\mathcal{F}_G [\text{Diagram 2}]) \quad /// G$$




- So, η_G is a 2d TQFT valued in holomorphic symplectic varieties.
- The rest of the talk is spent in describing what is known about η_G via stringy analysis.
- *Warning:* A common way to get a hyperkähler mfd. from a punctured Riemann surface is to consider the moduli space of Hitchin system. But η_G is not this. Rather, η_G is morally dual to the Hitchin system.

- The source category is the category of 2-bordisms.
- Objects are one-dimensional manifolds.
- Morphisms are cobordisms.
- Properties saying things like



- The target category is the category of *affine holomorphic symplectic varieties with Hamiltonian group action.*
- Objects are semisimple algebraic groups.
- Unit is the trivial group
- Multiplication of objects is just the Cartesian product $G \times G'$

- $\text{Hom}(G, G')$ is given by the set of affine holomorphic symplectic varieties with Hamiltonian action of $G \times G'$, together with a \mathbb{C}^\times action s.t. $\psi_t^*(\omega) = t^{-2}\omega$
- A typical example is T^*M and quiver varieties
- For $X \in \text{Hom}(G, G')$ and $Y \in \text{Hom}(H, H')$,
 $X \times Y \in \text{Hom}(G \times G', H \times H')$

- For $X \in \text{Hom}(G', G)$ and $Y \in \text{Hom}(G, G'')$, their composition $YX \in \text{Hom}(G', G'')$ is the *holomorphic symplectic quotient*

$$YX = \{\mu(X) = \mu(Y)\} / G$$

where μ is the Hamiltonian of G .

- This makes it a *symmetric monoidal category*.

- The identity in $\text{Hom}(G, G)$ is T^*G .

$$T^*G \simeq G \times \mathfrak{g} \ni (g, x)$$

Hamiltonian of the right action is x itself.

Therefore,

$$\begin{aligned} (T^*G)Y &= \{(g, x, y) \in G \times \mathfrak{g} \times Y \mid x = \mu(y)\} / G \\ &= Y. \end{aligned}$$

- The functor η_G is from the category of 2-bordisms to the category of hol. symplectic varieties with Hamiltonian actions.
- As for objects, we have

$$\eta_G[\emptyset] = G$$

- This was easy. The fun is in the morphisms.

- Two general properties are

$$\eta_G[\text{cylinder}] = T^*G$$

$$\eta_G[\text{cone}] = G \times S_\nu \subset G \times \mathfrak{g} \simeq T^*G$$

Here,

$$S_\nu = \{\nu + v \in \mathfrak{g} \mid [\nu^*, v] = 0\}$$

is the *Slodowy slice* at a nilpotent element v ;

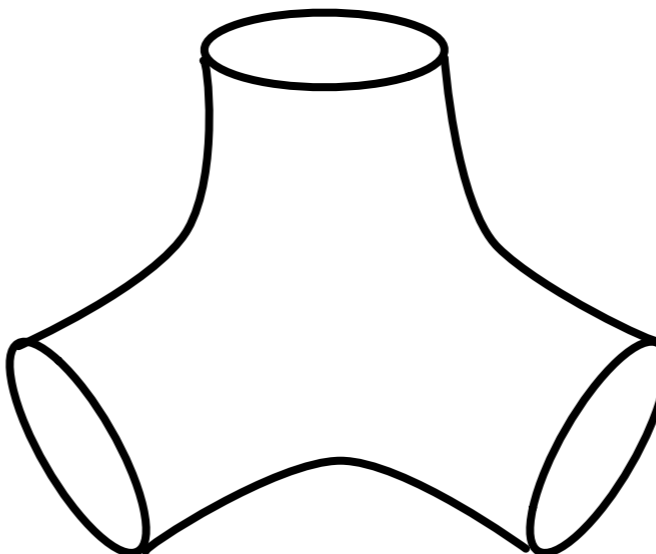
S_n is the one at a regular nilpotent element n .

- The problem is that

$$T_G = \eta_G \left[\text{[Diagram of a genus-1 surface with three boundary components]} \right]$$

is not known in general.

But it should be a marvelous manifold.

$$T_G = \eta_G \left[\text{Diagram} \right]$$


should have three G actions $\alpha_{1,2,3}(g) : T_G \rightarrow T_G$

and $\sigma : T_G \rightarrow T_G$ for $\sigma \in \mathfrak{S}_3$

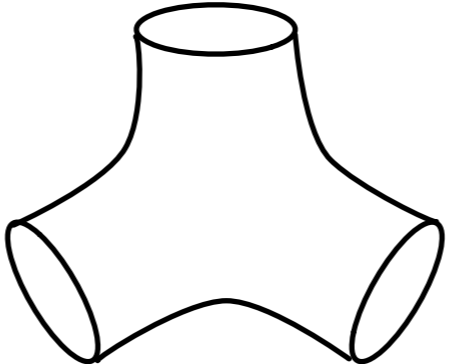
such that $\sigma \circ \alpha_i = \alpha_{\sigma(i)} \circ \sigma$

Its dimension is given by

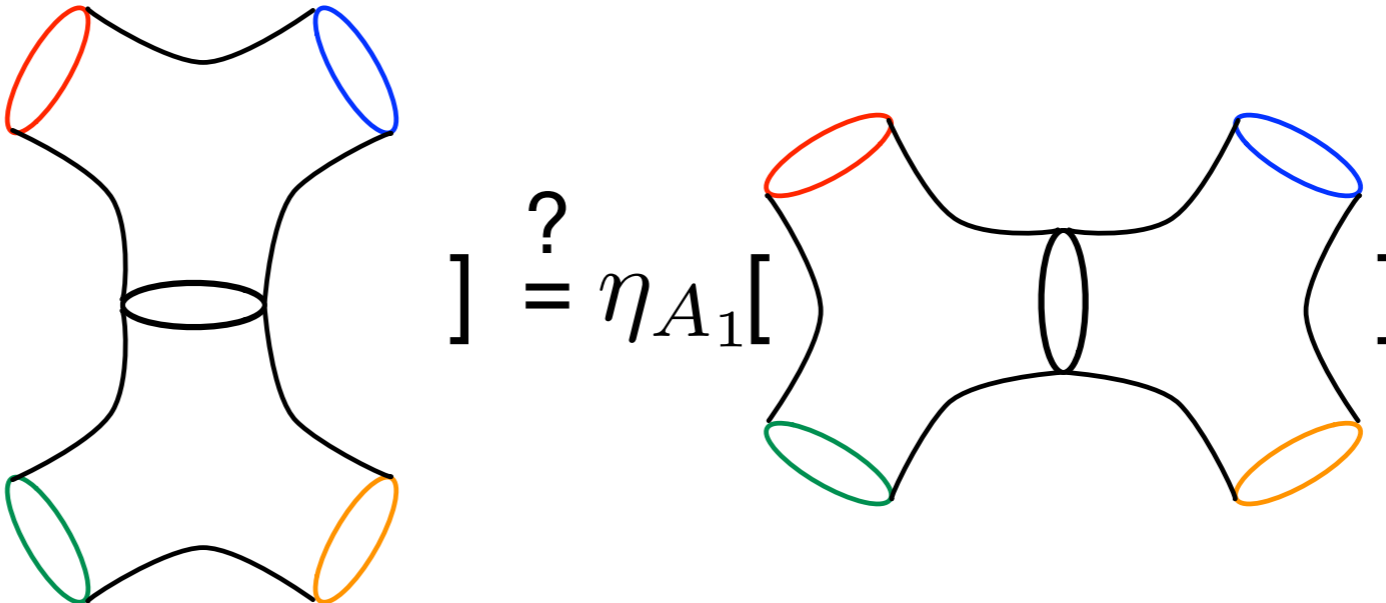
$$\dim_{\mathbb{C}} T_G = 2 \operatorname{rank} G + 3 \dim_{\mathbb{C}} \mathcal{N}$$

where \mathcal{N} is the nilpotent cone in \mathfrak{g}

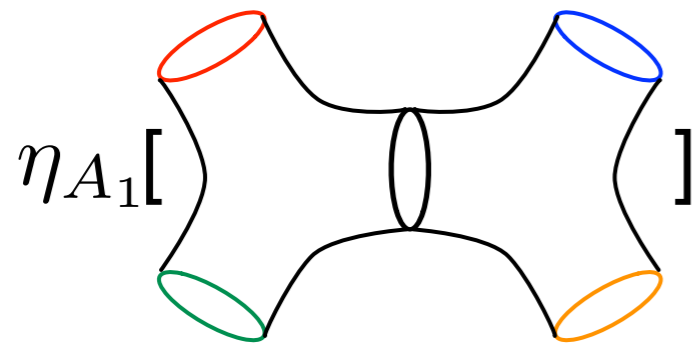
- When $G=A_1$,

$$T_{A_1} = \eta_{A_1} \left[\text{trinion} \right] = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$$


- The “associativity” is not completely obvious:

$$\eta_{A_1} \left[\text{trinion with colored boundaries} \right] \stackrel{?}{=} \eta_{A_1} \left[\text{trinion with colored boundaries} \right]$$


- It turns out that



$$= (\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2) \times (\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2) // SL(2, \mathbb{C})$$

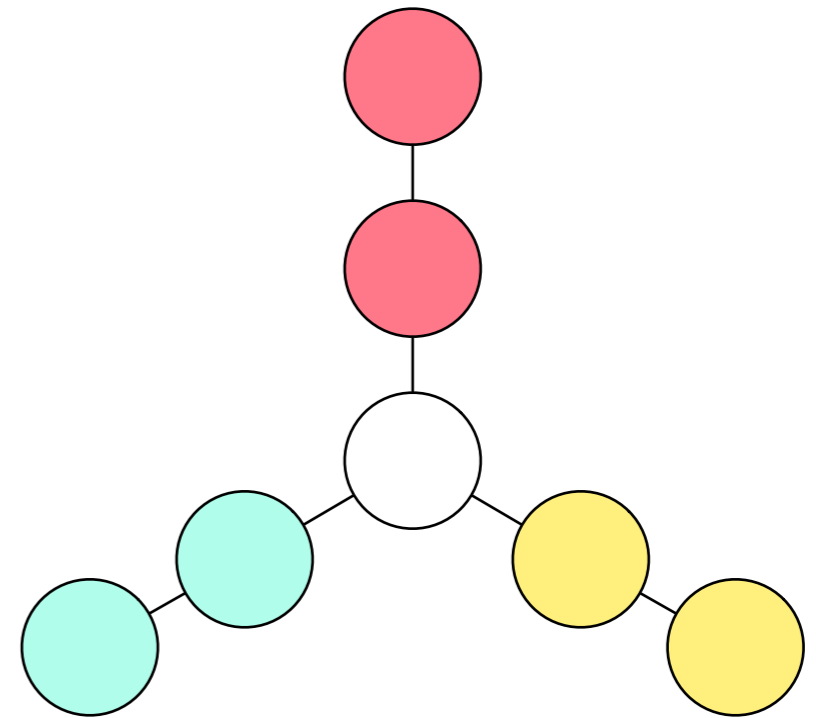
is the ADHM description of $SO(8)$ 1-instanton moduli space.

- Outer automorphism S_3 of $SO(8)$ guarantees the associativity.

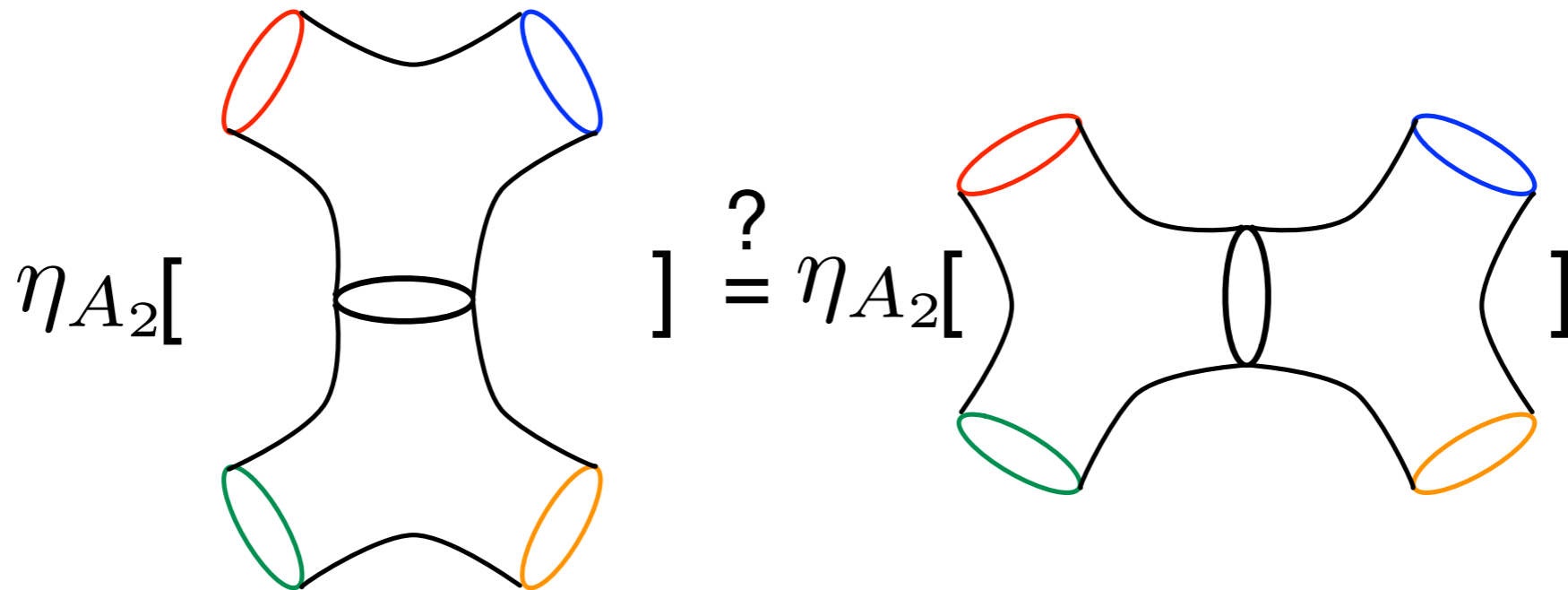
- Let's move on to $G=A_2$.

η_{A_2} [] = minimal nilpotent orbit of E_6

$$SL(3) \times SL(3) \times SL(3) \subset E_6$$



- **Associativity?**



Please prove it! It only takes finite amount of time.

- To describe known properties of η_G for general G , we need to consider Bielawski's slicing.
- Let us introduce, for $\rho : \mathfrak{sl}(2) \rightarrow \mathfrak{g}$

$$\eta_G \left[\text{cone}(\rho) \right] = G \times S_{\rho(e)} \subset G \times \mathfrak{g} \simeq T^*G$$

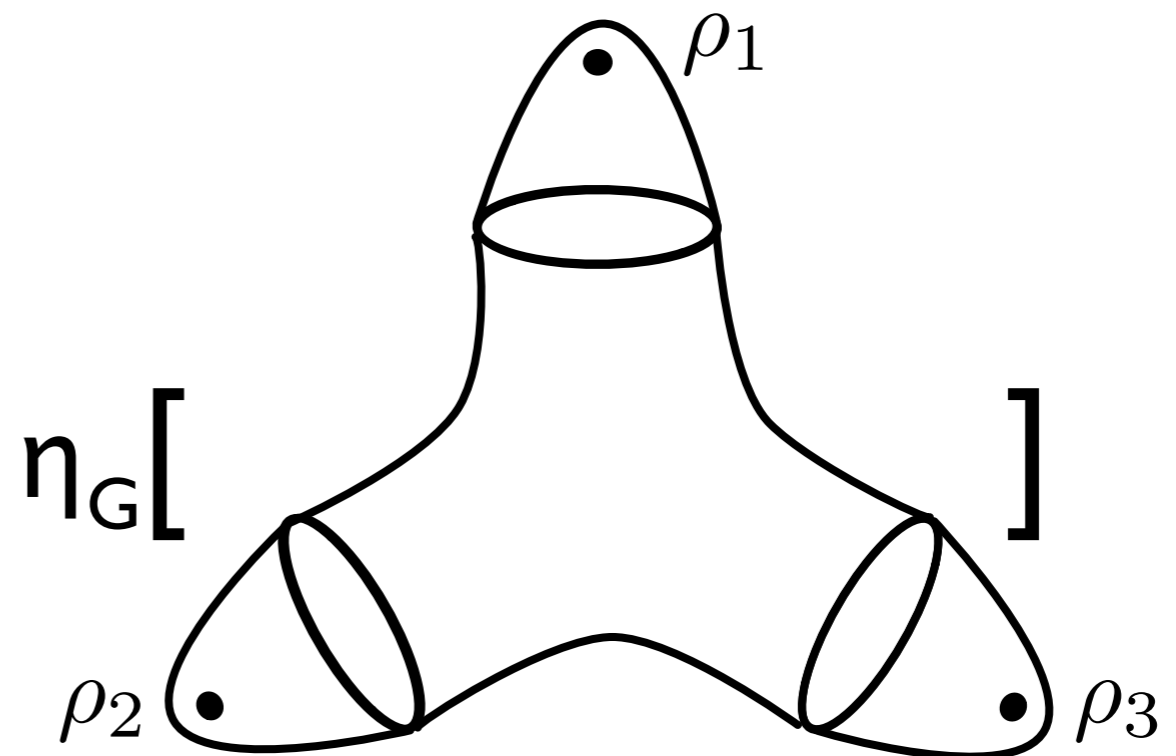
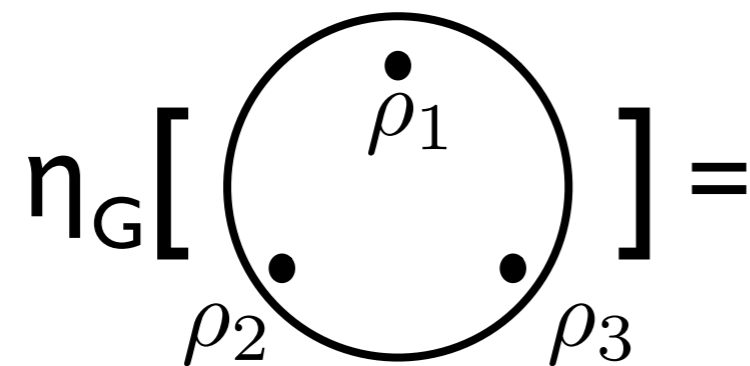
where $S_{\rho(e)} = \{ \rho(e) + v \in \mathfrak{g} \mid [\rho(f), v] = 0 \}$

is the Slodowy slice at $\rho(e)$.

(e, f, h) is the $\mathfrak{sl}(2)$ triple.

- Using these caps, we consider

$$T_G[\rho_1, \rho_2, \rho_3] =$$



- This is a hol. sympl. variety obtained by applying Bielawski's slicing to T_G .

- For A_{N-1} , $\rho : \mathfrak{sl}(2) \rightarrow \mathfrak{sl}(N)$
is characterized by a partition of N ,
with which ρ is identified.

$$T_{A_{N-1}}[(N-1, 1), (1^N), (1^N)] = V \otimes V^* \oplus V^* \otimes V$$

$$T_{A_{N-1}}\left[\left(\lfloor \frac{N+1}{2} \rfloor, \lfloor \frac{N}{2} \rfloor\right), \left(\lfloor \frac{N}{2} \rfloor, \lfloor \frac{N-1}{2} \rfloor, 1\right), (1^N)\right] = \wedge^2 V \oplus \wedge^2 V^* \oplus V \otimes \mathbb{C}^2 \oplus V^* \otimes \mathbb{C}^2$$

where $V = \mathbb{C}^N$.

These are just symplectic vector spaces.

- More surprising properties are that

$$T_{A_{3k-1}} [(k, k, k), (k, k, k), (k, k, k)]$$

$$T_{A_{4k-1}} [(k, k, k, k), (k, k, k, k), (2k, 2k)]$$

$$T_{A_{6k-1}} [(k, k, k, k, k, k), (2k, 2k, 2k), (3k, 3k)]$$

are the framed centered k-instanton moduli spaces of $E_{6,7,8}$, respectively.

- So, $T_G = \eta_G [\text{trinion}]$

are very intriguing holomorphic symplectic varieties which satisfy associativity

$$\eta_G [\text{trinion with 4 colored boundaries}] = \eta_G [\text{trinion with 4 colored boundaries}]$$

and which can give exceptional instanton moduli spaces after slicing.

- Summarizing, the properties of the functor η_G from 2-bordisms to hol. sympl. varieties were described .
- This is conjectural for $G \neq A_1$.
Please construct it.
- The full set of axioms and known properties will be available soon on the arXiv.
- As a prize, I will offer a nice dinner at the Sushi restaurant in the University of Tokyo campus where the IPMU is.