On 2d TQFT whose values are hyperkähler cones

Yuji Tachikawa (IAS & IPMU)

at String-Math 2011, U. Penn

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On 2d TQFT whose values are holomorphic symplectic varieties

Yuji Tachikawa (IAS & IPMU)

in collaboration with **Greg Moore** (Rutgers)

at String-Math 2011, U. Penn

- I'll give a *mathematical reformulation* of the *known results in string theory* literature.

Argyres-*Seiberg*, *Argyres*-*Wittig*,
Gaiotto-*Witten*,
Gaiotto-*Neitzke*-*YT*,
Benvenuti-*Benini*-*YT*,
Chacaltana-*Distler*, ...

- I'll start in a *stringy* language, and gradually make it *mathematically more precise*.

- In early 2009, Davide Gaiotto came up with a great idea.

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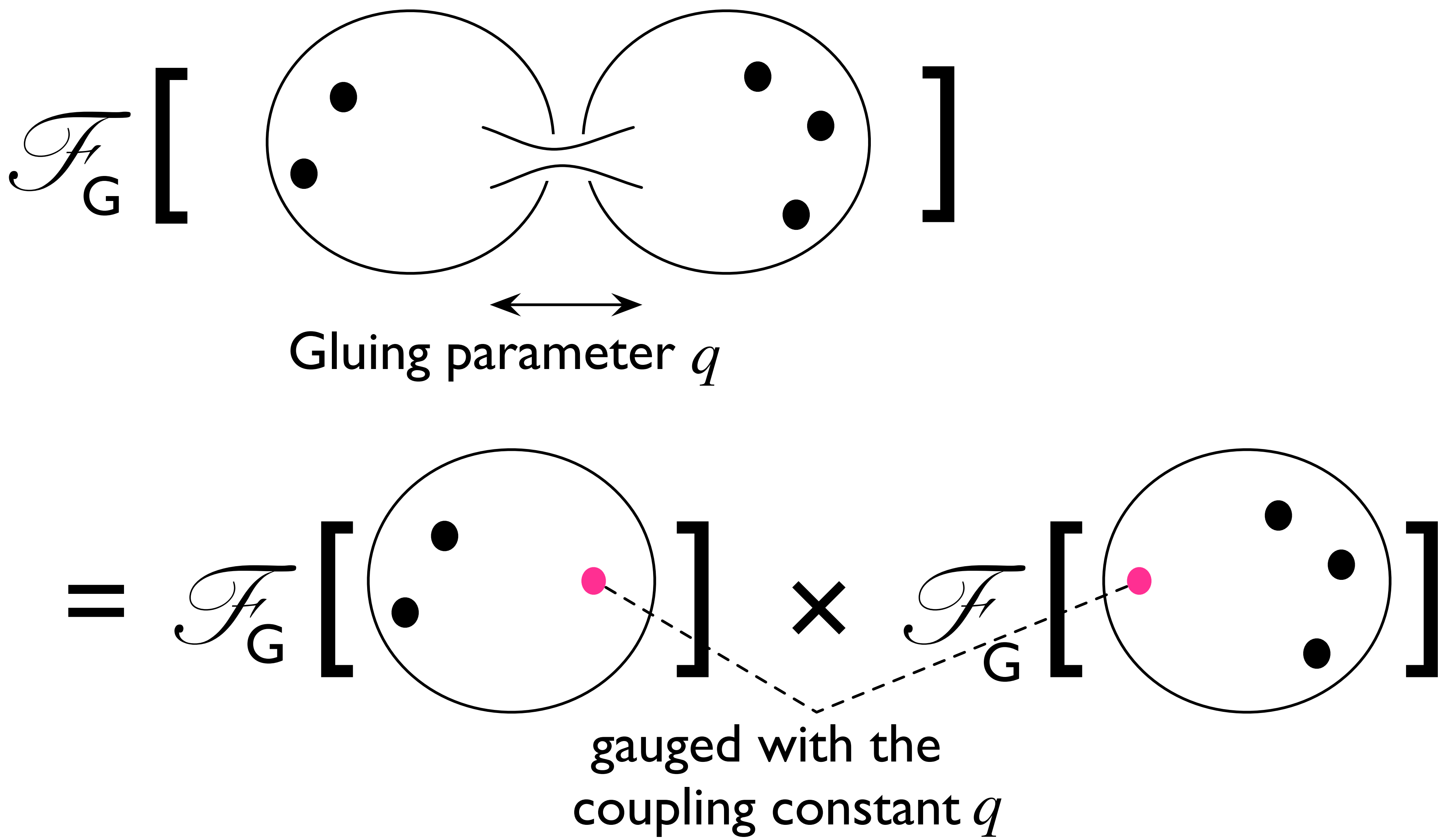
Wrap a 6d $N=(2,0)$ theory on a Riemann surface C with punctures.

This gives a 4d $N=2$ theory depending on the complex structure of C.

Then, the 4d $N=2$ theory decomposes nicely into two, when the surface C is pinched into two surfaces.

- Mathematically, there should be a functor $\mathscr{F}_G$ from the *category of Riemann surfaces* to the *category of 4d N=2 theories* for $G=A_n,D_n,E_n$



- The problem is that the *category of 4d N=2 theories* is yet to be defined.
- However, some part of it can still be formulated rigorously.

Take a Riemannian manifold M

choose an Abelian group A

$H^*(M,A)$

: A-module

dim(M)

: +ve integer

vol(M)

: real number

Various objects associated functorially, depending on various amount of structures on M

Take a 4d N=2 theory T

choose a 4-mfd X

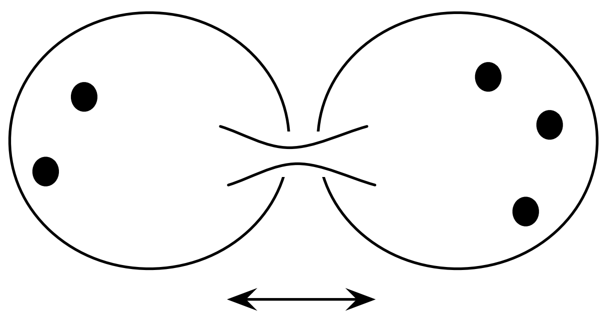
Z(T,X)
: complex number

Coulomb(T)
: special Kähler mfd

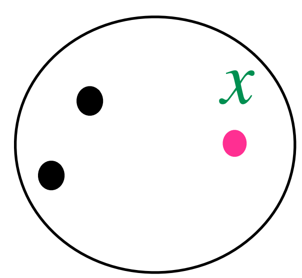
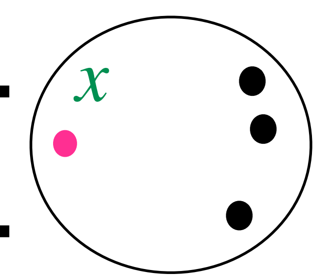
Higgs(T)
: hyperkähler mfd

Various objects associated functorially,
depending on various amount of structures on T

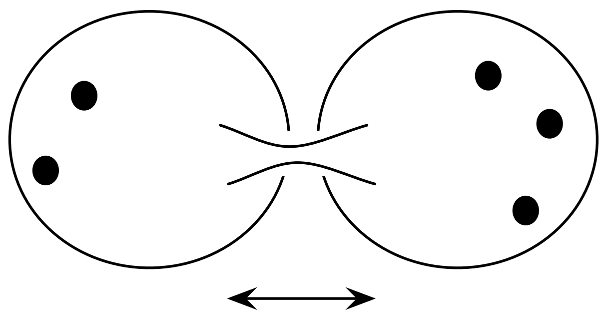
- Let us compose the Gaiotto functor $\mathscr{F}_G$ from the *category of Riemann surfaces* to the *category of 4d N=2 theories* with the functor $Z(. , S^4)$.

$Z(\mathscr{F}_G[$  $], S^4)$

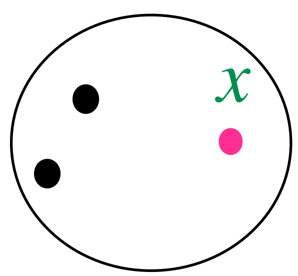
Gluing parameter q

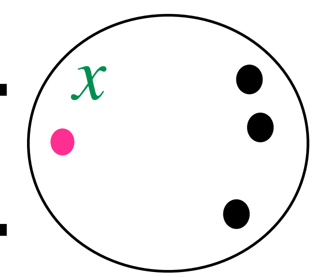
$= \int_{-\infty}^{\infty} q^{-x^2}\, Z(\mathscr{F}_G[$  $], S^4)\; Z(\mathscr{F}_G[$  $], S^4)\, dx$

- The functor $Z\mathscr{F}_G$ is
  from the *category of Riemann surfaces*
  to the *category of vector spaces.*
  i.e. this is a 2d conformal field theory.

$Z(\mathscr{F}_G[$  $],S^4)$

Gluing parameter q

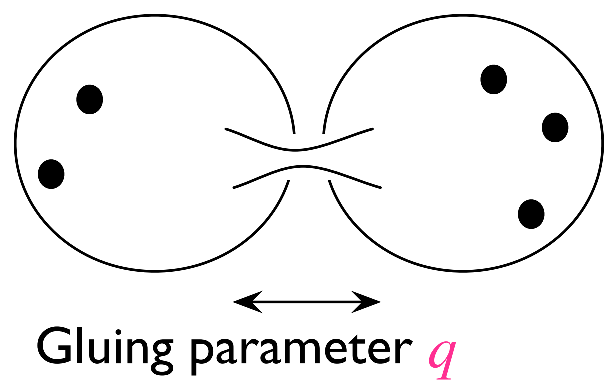
$= \int_{-\infty}^{\infty} q^{-x^2}\, Z(\mathscr{F}_G[$ 

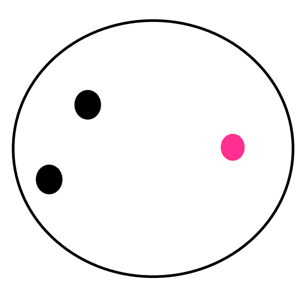
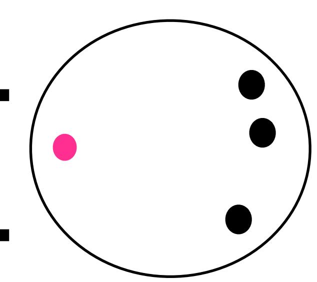
$],S^4)\; Z(\mathscr{F}_G[$ 

$],S^4)\; dx$

- This is the essence of the AG❄ correspondence.

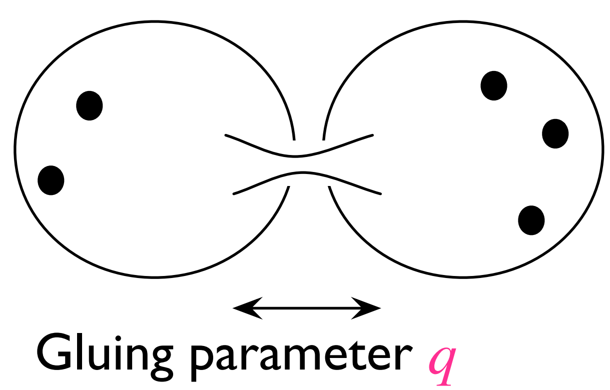
- Let us compose the Gaiotto functor $\mathscr{F}_G$
  from the *category of Riemann surfaces*
  to the *category of 4d N=2 theories*
  with the functor Higgs(.)
  to the category of hyperkähler manifolds.

Higgs($\mathscr{F}_G$ [])

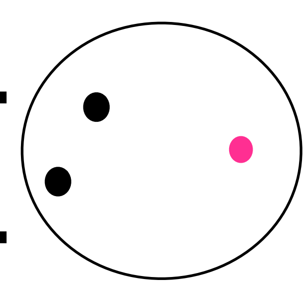
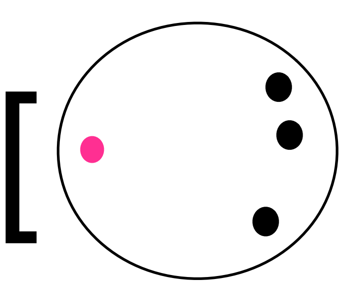
= Higgs($\mathscr{F}_G$ []) × Higgs($\mathscr{F}_G$ []) /// G

- This is independent of q as holomorphic symplectic varieties.

- η_G = Higgs$\mathscr{F}_G$ is a functor
  from the *category of 2-cobordisms*
  to the *category of hol. sympl. varieties.*

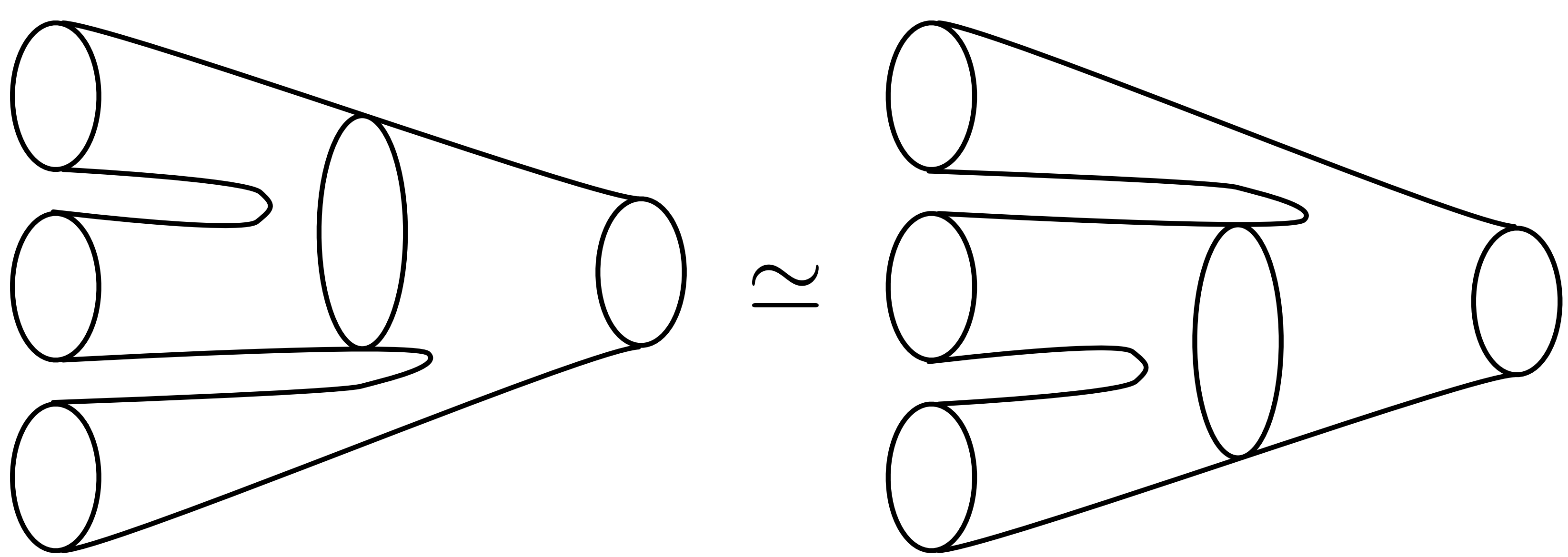
Higgs($\mathscr{F}_G$ [

 ])

= Higgs($\mathscr{F}_G$ []) × Higgs($\mathscr{F}_G$ []) /// G

- So, η_G is a 2d TQFT valued in holomorphic symplectic varieties.
- The rest of the talk is spent in describing what is known about η_G via stringy analysis.
- *Warning:* A common way to get a hyperkähler mfd. from a punctured Riemann surface is to consider the moduli space of Hitchin system. But η_G is not this. Rather, η_G is morally dual to the Hitchin system.

- The source category is the category of 2-bordisms.
- Objects are one-dimensional manifolds.
- Morphisms are cobordims.
- Properties saying things like



- The target category is the category of *affine holomorphic symplectic varieties with Hamiltonian group action.*

- Objects are semisimple algebraic groups.

- Unit is the trivial group

- Multiplication of objects is just the Cartesian product G x G’

- Hom(G,G') is given by the set of affine holomorphic symplectic varieties with Hamiltonian action of G x G', together with a $\mathbb{C}^\times$ action s.t. $\psi_t^*(\omega) = t^{-2}\omega$

- A typical example is T^*M and quiver varieties

- For X $\in$ Hom(G,G') and Y $\in$ Hom (H,H'),

  X x Y $\in$ Hom(G x G', H x H')

- For X∈Hom(G',G) and Y∈Hom(G,G''), their composition YX∈Hom(G',G'') is the *holomorphic symplectic quotient*

$$YX = \{\mu(X) = \mu(Y)\}/G$$

where μ is the Hamiltonian of G.

- This makes it a *symmetric monoidal category*.

- The identity in Hom(G,G) is T*G.

$$T^*G \simeq G \times \mathfrak{g} \ni (g, x)$$

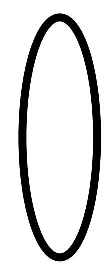
Hamiltonian of the right action is x itself.

Therefore,

$$\begin{aligned}(T^*G)Y &= \{(g, x, y) \in G \times \mathfrak{g} \times Y \mid x = \mu(y)\}/G \\ &= Y.\end{aligned}$$

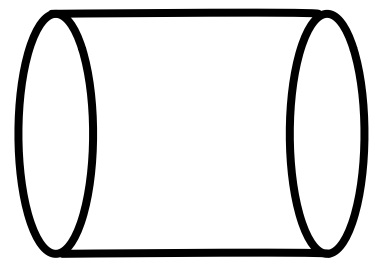
- The functor η_G is from the category of 2-bordisms to the category of hol. symplectic varieties with Hamiltonian actions.

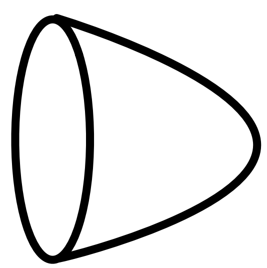
- As for objects, we have

$\eta_G[$  $] = G$

- This was easy. The fun is in the morphisms.

- Two general properties are

η_G[] = T^*G

η_G[] = $G \times S_n \subset G \times \mathfrak{g} \simeq T^*G$

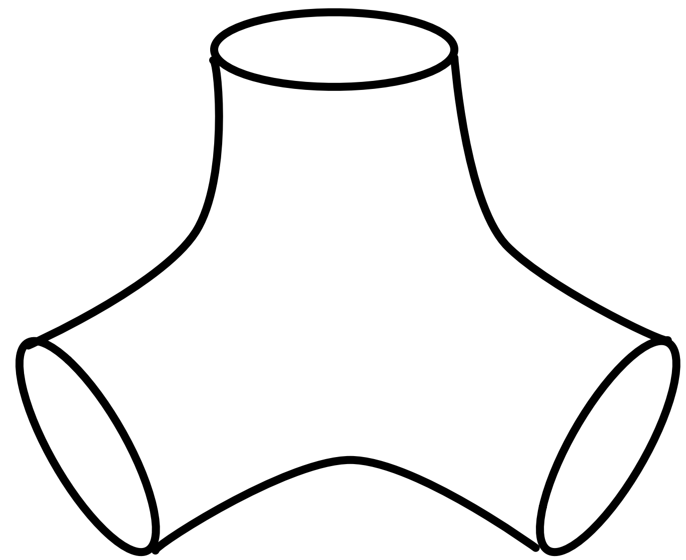
Here,

$$S_\nu = \{\nu + v \in \mathfrak{g} \mid [\nu^*, v] = 0\}$$

is the *Slodowy slice* at a nilpotent element *ν* ;

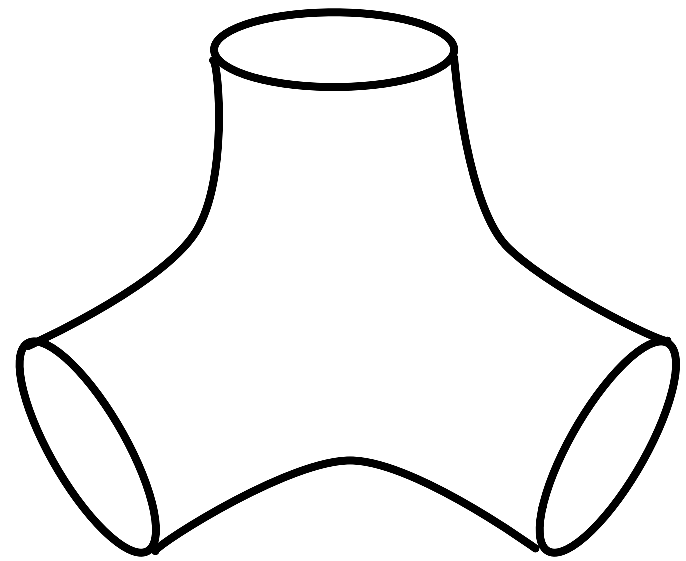
S_n is the one at a regular nilpotent element *n*.

- The problem is that

$$T_G = \eta_G\Big[$$  $$\Big]$$

is not known in general.

But it should be a marvelous manifold.

T_G = η_G[]

should have three G actions $\alpha_{1,2,3}(g) : T_G \to T_G$

and $\sigma : T_G \to T_G$ for $\sigma \in \mathfrak{S}_3$

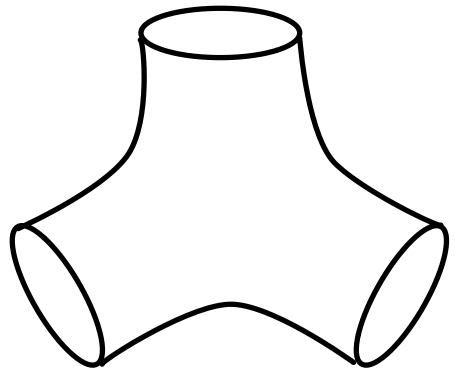
such that $\sigma \circ \alpha_i = \alpha_{\sigma(i)} \circ \sigma$

Its dimension is given by

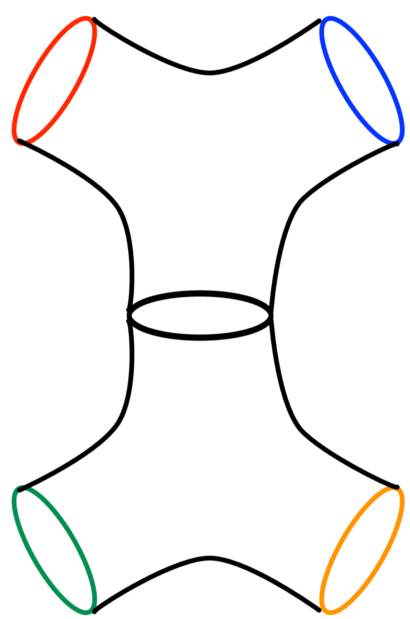
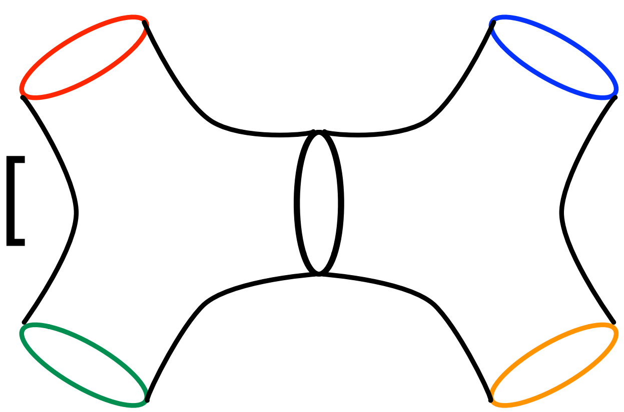
$$\dim_{\mathbb{C}} T_G = 2\,\mathrm{rank} G + 3 \dim_{\mathbb{C}} \mathcal{N}$$

where *N* is the nilpotent cone in $\mathfrak{g}$

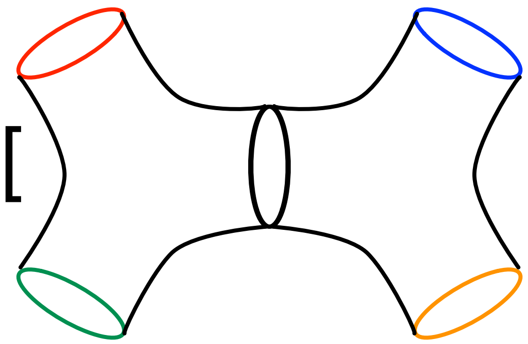
- When G=A$_1$,

$T_{A_1} = \eta_{A_1}[$  $] = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$

- The "associativity" is not completely obvious:

$\eta_{A_1}[$  $] \stackrel{?}{=} \eta_{A_1}[$  $]$

- It turns out that

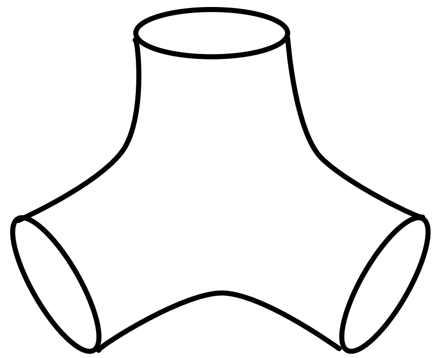
$\eta_{A_1}[$  $]$

$$= (\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2) \times (\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2)//SL(2,\mathbb{C})$$

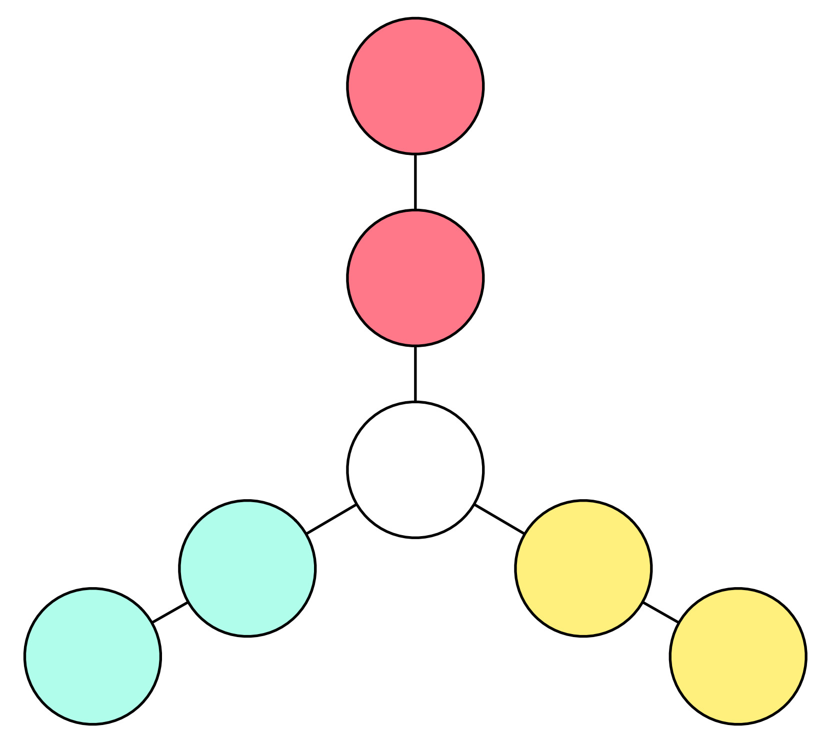
  is the ADHM description of SO(8) 1-instanton moduli space.

- Outer automorphism S_3 of SO(8) guarantees the associativity.

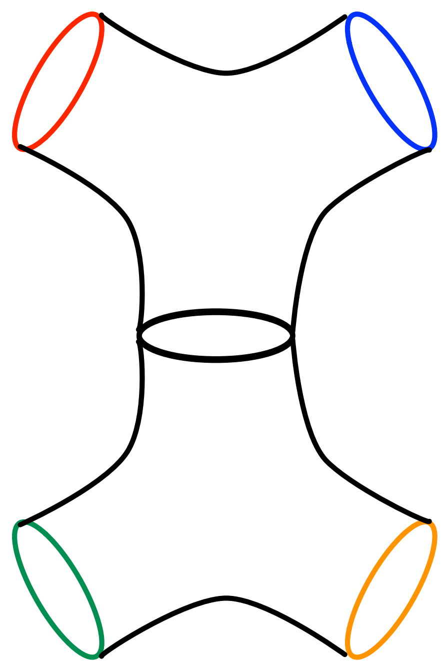
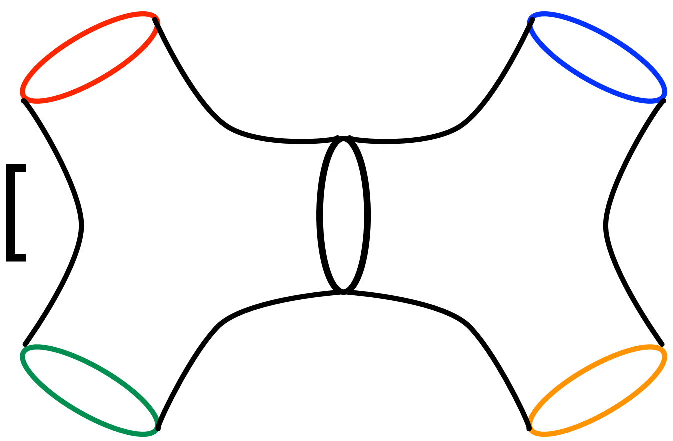
- Let's move on to G=A_2 .

η_{A_2} [] = minimal nilpotent orbit of E_6

$$SL(3) \times SL(3) \times SL(3) \subset E_6$$



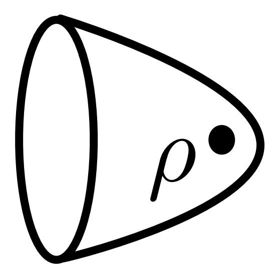
- Associativity?

$\eta_{A_2}[$  $] \stackrel{?}{=} \eta_{A_2}[$  $]$

Please prove it! It only takes finite amount of time.

- To describe known properties of η_G for general G, we need to consider Bielawski's slicing.

- Let us introduce, for $\rho : \mathfrak{sl}(2) \to \mathfrak{g}$

η_G[

] = $G \times S_{\rho(e)} \subset G \times \mathfrak{g} \simeq T^*G$

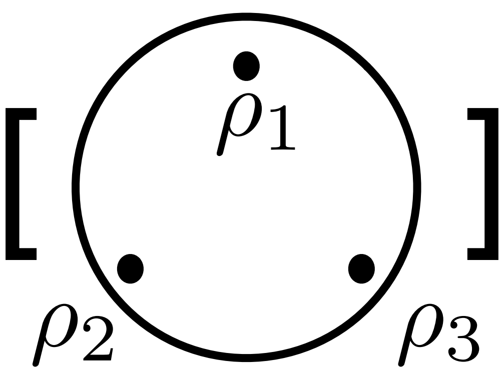
where $S_{\rho(e)} = \{\rho(e) + v \in \mathfrak{g} \mid [\rho(f), v] = 0\}$

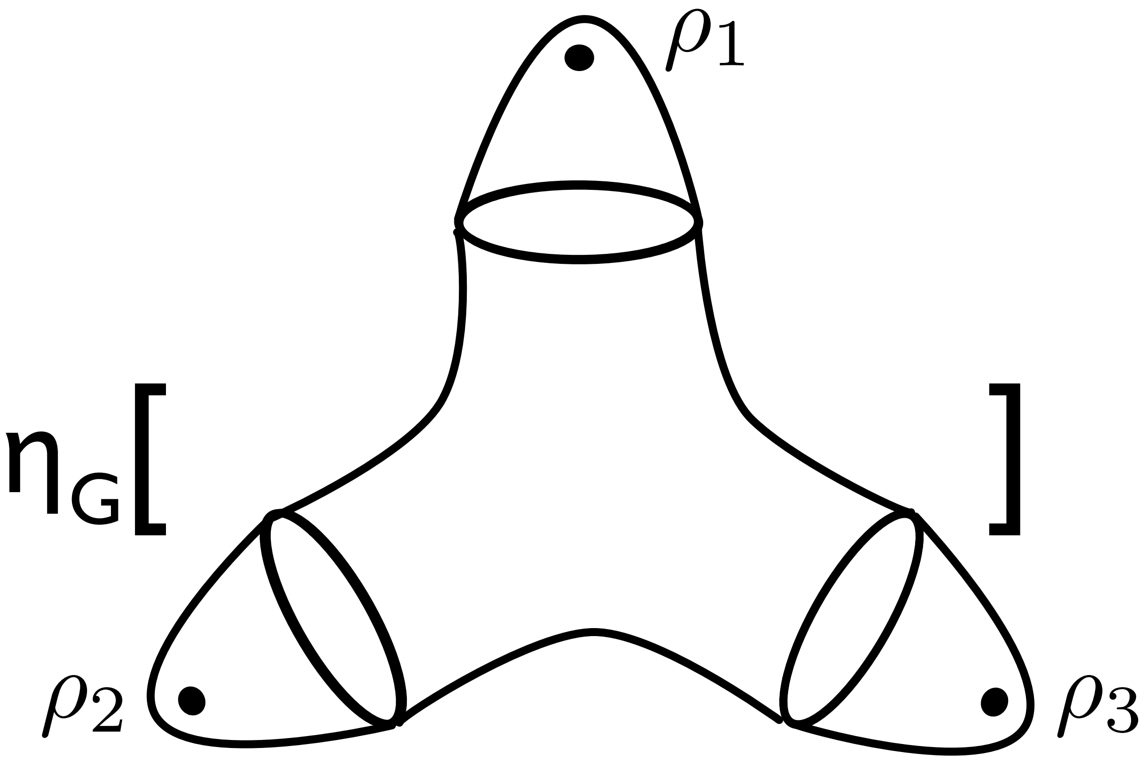
is the Slodowy slice at $\rho(e)$.

(e, f, h) is the sl(2) triple.

- Using these caps, we consider

$T_G[\rho_1, \rho_2, \rho_3] =$

$\eta_G[$ 

$] = \eta_G[$ 

$]$

- This is a hol. sympl. variety obtained by applying Bielawski's slicing to T_G .

- For A_{N-1}, $\rho : \mathfrak{sl}(2) \to \mathfrak{sl}(N)$ is characterized by a partition of N, with which ρ is identified.

$$T_{A_{N-1}}[(N-1,1),(1^N),(1^N)] = V \otimes V^* \oplus V^* \otimes V$$

$$T_{A_{N-1}}[(\lfloor \frac{N+1}{2} \rfloor, \lfloor \frac{N}{2} \rfloor),(\lfloor \frac{N}{2} \rfloor, \lfloor \frac{N-1}{2} \rfloor, 1),(1^N)] = \wedge^2 V \oplus \wedge^2 V^* \oplus V \otimes \mathbb{C}^2 \oplus V^* \otimes \mathbb{C}^2$$

where $V = \mathbb{C}^N$.

These are just symplectic vector spaces.

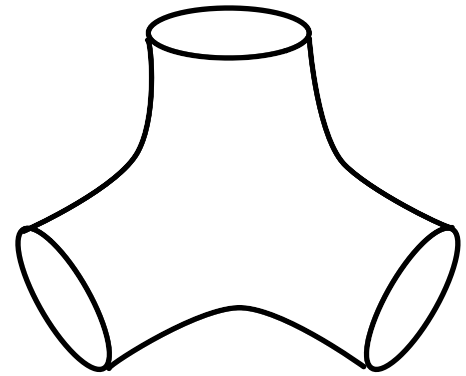
- More surprising properties are that

$$T_{A_{3k-1}}[(k,k,k),(k,k,k),(k,k,k)]$$

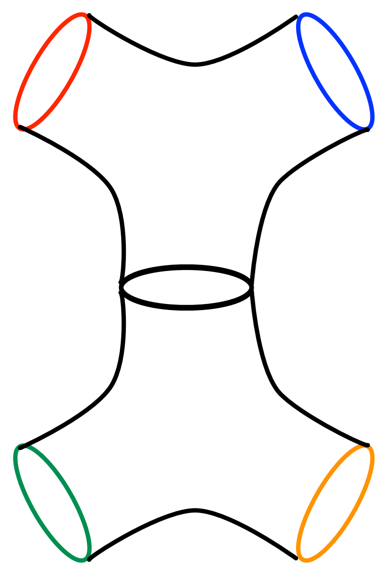
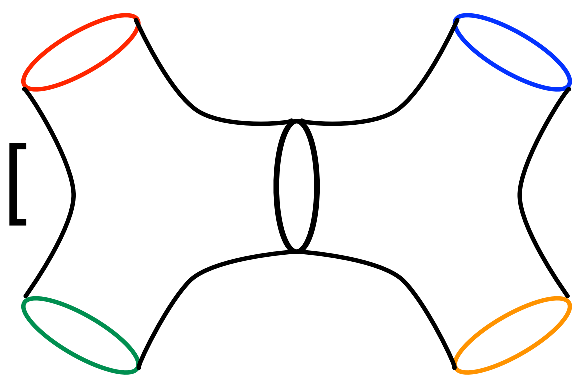
$$T_{A_{4k-1}}[(k,k,k,k),(k,k,k,k),(2k,2k)]$$

$$T_{A_{6k-1}}[(k,k,k,k,k,k),(2k,2k,2k),(3k,3k)]$$

are the framed centered k-instanton moduli spaces of $E_{6,7,8}$, respectively.

- So, $T_G = \eta_G[$  $]$

are very intriguing holomorphic symplectic varieties which satisfy associativity

$\eta_G[$  $] = \eta_G[$  $]$

and which can give exceptional instanton moduli spaces after slicing.

- Summarizing, the properties of the functor η_G from 2-bordisms to hol. sympl. varieties were described .

- This is conjectural for $G \neq A_1$.
Please construct it.

- The full set of axioms and known properties will be available soon on the arXiv.

- As a prize, I will offer a nice dinner at the Sushi restaurant in the University of Tokyo campus where the IPMU is.