# Vanishing Chiral Algebras and Höhn–Stolz Conjecture

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# Introduction

The chiral algebras of $(0, 2)$ sigma models are

- generalizations of quantum cohomology rings
- related to chiral differential operators, $(0, 2)$ mirror symmetry, geometric Langlands, etc.

I will discuss:

- the chiral algebras can vanish
- implications for the geometry of loop spaces
- esp. Höhn–Stolz conjecture in the Kähler case.

# Höhn–Stolz conjecture

$M$: closed, string (spin & $p_1(M)/2 = 0$) manifold

If Ric > 0, then the Witten genus

$$\boxed{\phi_W(M) = 0.}$$

[Stolz, Math. Ann. **304** (1996) 785; Höhn, unpublished]

Here

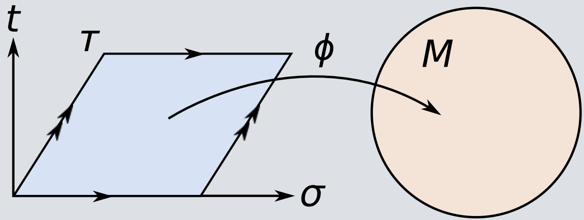
$$\phi_W(M) = \prod_{m=1}^{\infty}(1-q^m)^{\dim M}\sum_{n=0}^{\infty} q^n \mathrm{index}(D \otimes V_n),$$

$D$: Dirac op; $V_0 = 1$, $V_1 = TM$, $V_2 = TM \oplus S^2TM$, $\ldots$.

✓complete intersections, $G/H$, ...

# $(0,1)$ sigma model

Sigma model on torus




Add fermions; then it has $(0,1)$ SUSY:

$$Q = Q^*, \quad Q^2 = H - P \quad (H = -\partial_t,\ P = -i\partial_\sigma).$$

For the theory to be consistent, $M$ must be string.

Partition function

$$Z = \phi_W(M)/\eta(q)^{\dim M}, \quad q = e^{i\tau}.$$

# Loop-space viewpoint

View as quantum mechanics on $\mathcal{L}M = \{S^1 \to M\}$:

| QFT | loop space |
| --- | --- |
| $\{\psi^a_\sigma, \psi^b_{\sigma'}\} = \delta^{ab}\delta_{\sigma\sigma'}$ | Clifford algebra |
| states | spinors |
| parity | chirality |
| $Q$ | Dirac operator |
| SUSY states | harmonic spinors |

$Z$ is the $S^1$-equivariant index of Dirac operator:

$$Z = \operatorname{str} q^P,$$

where str is over {harmonic spinors on $\mathcal{L}M$}.

# Loop-space Lichnerowicz

Lichnerowicz: no harmonic spinors if $R > 0$.

Analogy with the finite-dimensional case:

| $M$ | $\mathcal{L}M$ |
|---|---|
| spin | string |
| $R > 0$ | $\mathrm{Ric} > 0$ |
| $\widehat{A}(M) = 0$ | $\phi_W(M) = 0$? |

Stolz's idea: "loop-space Lichnerowicz"

$$\boxed{\text{no harmonic spinors on } \mathcal{L}M \text{ if } \mathrm{Ric} > 0}$$

will imply the Höhn–Stolz conjecture.

# Kähler case: $(0,2)$ sigma model

If $M$ is Kähler, $D = \mathcal{D} + \mathcal{D}^*$ with $\mathcal{D}^2 = 0$, so we can consider the $\mathcal{D}$-cohomology (spinor cohomology).

Similarly, $Q = \mathcal{Q} + \mathcal{Q}^*$ and we have $(0,2)$ SUSY:

$$\mathcal{Q}^2 = 0, \qquad \{\mathcal{Q}, \mathcal{Q}^*\} = H - P.$$

The $\mathcal{Q}$-cohomology of states

$$H^\bullet_{\mathcal{Q}} \cong \{\text{harmonic spinors on } \mathcal{L}M\}.$$

Loop-space Lichnerowicz in the Kähler case will be

$$\boxed{H^\bullet_{\mathcal{Q}} = 0 \text{ if } c_1 > 0.}$$

# Chiral algebra from $(0, 2)$ SUSY

Now consider the $\mathcal{Q}$-cohomology of operators.

Its elements

- vary holomorphially:

$$\partial_{\bar{z}}\mathcal{O} = [H - P, \mathcal{O}] = [\mathcal{Q}, \dots]$$

- have operator product expansions (OPE):

$$[\mathcal{O}_i(z)] \cdot [\mathcal{O}_j(w)] \sim c_{ij}{}^k(z - w)[\mathcal{O}_k(w)]$$

so define a chiral algebra $\mathcal{A}$, an OPE algebra of holomorphic fields.

# Chiral differential operators

Ignoring instantons,

$$\mathcal{A} = H^\bullet(\mathcal{D}_M).$$

$\mathcal{D}_M$: a sheaf of CDO. [Witten, hep-th/0504078]

Locally, $\mathcal{D}_M$ is a vertex alg known as the $\beta\gamma$ system:

$$\beta_i(z)\gamma^j(w) \sim \frac{\delta_i^j}{z-w}.$$

Some properties:

- $H^\bullet(\mathcal{D}_{G/B})$ is a $\hat{g}$-module of critical level.
- The energy-momentum tensor $L \notin H^\bullet$ if $c_1 \neq 0$.

# Vanishing theorem

Witten's prediction: for $M = \mathbb{P}^1$, instantons make

$$1 = 0$$

and the chiral algebra vanishes. [Witten, hep-th/0504078]

Verified. [Tan & JY, 0801.4782; Arakawa & Malikov, 0911.0922]

More generally:

If $\exists \mathbb{P}^1 \subset M$ with trivial normal bundle, then $\mathcal{A} = 0$.

Ex. $G/B$. [JY, 1002.0028; Frenkel–Losev–Nekrasov]

# SUSY breaking

$H^\bullet_{\mathcal{Q}}$ is a module over $\mathcal{A}$:

$$[\mathcal{O}] \cdot [|\Psi\rangle] = [\mathcal{O}|\Psi\rangle].$$

If the chiral algebra vanishes, then

$$[|\Psi\rangle] = [1] \cdot [|\Psi\rangle] = 0.$$

So the $\mathcal{Q}$-cohomology of states vanishes as well.

The spinor cohomology of $\mathcal{L}M$ is zero, there are no harmonic spinors on $\mathcal{L}M$, and

$$\phi_W(M) = 0.$$

# Question

$G/B$ has $c_1 > 0$ and $\mathcal{A} = 0$. Do we have:

$$\boxed{\text{the chiral algebra vanishes if } c_1 > 0}\,?$$

This will imply the Kähler loop-space Lichnerowicz.

Renormalization group ($\sim$ Ricci) flow argument:

1. Rescale the 2d metric $g \to tg$. $\mathcal{A}$ is invariant.
2. For $t \sim 0$, quantum effects is small if $c_1 > 0$.
3. Show $\mathcal{A}$ admits no energy-momentum tensor.
4. As $t \to \infty$, the theory flows to a SCFT, so $L \in \mathcal{A}$.
5. The SCFT is trivial, with $\mathcal{A} = 0$.

# Summary

"Loop-space Lichnerowicz" implies Höhn–Stolz.

For $M$ Kähler, no harmonic spinors on $\mathcal{L}M$ iff $H_{\mathcal{Q}} = 0$.

$\mathcal{A} = H^{\bullet}(\mathcal{D}_M)$ ignoring instantons.

$\mathcal{A} = 0$ in some cases in the presence of instantons.

$\mathcal{A} = 0$ implies $H_{\mathcal{Q}} = 0$.

$\mathcal{A} = 0$ if $c_1 > 0$?