Non-Kähler Calabi-Yau Manifolds

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String and math have had a very close interaction over the past thirty years. It has been extremely fruitful and produced many beautiful results.

For example, mathematical research on Calabi-Yaus over the past two decades have been strongly motivated by string, and in particular, mirror symmetry.

Mirror symmetry started from the simple observation by Dixon, and Lerche-Vafa-Warner, around 1989, of a possible geometric realization of flipping the sign of a representation of the super-conformal algebra. Geometrically, it implied that Calabi-Yaus should come in pairs with the pair of Hodge numbers, $h^{1,1}$ and $h^{2,1}$, exchanged.

Shortly following the observation, Greene-Plesser gave an explicit construction of the mirror of the Fermat quintic using the orbifold construction. Furthermore, Candelas-de la Ossa-Green-Parkes found as a consequence of mirror symmetry a most surprising formula for counting rational curves on a general quintic.

The identification of the topological A and B model by Witten further inspired much rigorous mathematical work to justify various definitions and relations, such as the Gromov-Witten invariants, multiple cover formula, and other related topics. More works by Witten, Kontsevich and many others led to independent proofs by Givental and Lian-Liu-Yau of the Candelas *et al.* formula for the genus zero Gromov-Witten invariants in the mid-1990s.

The string prediction of the genus one Gromov-Witten invariants of Bershadsky-Cecotti-Ogurri-Vafa (BCOV) for the quintic was only proved by Zinger and Jun Li about five years ago.

Though we now know much about mirror symmetry, many important questions remain and progress continues to be made.

The higher genus $g \geq 2$ case is still mathematically not well-understood. In the celebrated work of BCOV (1993), a holomorphic anomaly equation for higher genus partition functions, F_g, was written down.

Yamaguchi and I in 2004 were able to show that F_g for $g \geq 2$ are polynomials of just five generators: $(V_1, V_2, V_3, W_1, Y_1)$. With the generators assigned the degree, $(1, 2, 3, 1, 1)$ respectively, then F_g is a quasi-homogeneous polynomial of degree $(3g - 3)$.

Huang-Klemm-Quackenbush were able to use this result to compute the partition function on the mirror quintic up to genus $g = 51$.

As noted by BCOV, the higher genus B-model partition can come from the quantization of Kodaira-Spencer gauge theory. Recently, Costello and my student Si Li have made significant progress. They gave a prescription for quantizing the Kodaira-Spencer theory and have successfully carried it out in the elliptic curve case.

Much of the work on mirror symmetry have been based on toric geometry. To go beyond toric cases, one need to study period integrals and the differential equations which govern them under complex structure deformation. In this regards, Bong Lian, my student Ruifang Song, and I have very recently been able to describe explicitly a Picard-Fuchs type differential system for Calabi-Yau complete intersections in a Fano variety or a homogenous space.

From the geometric perspective, Strominger-Yau-Zaslow gave a T-duality explanation of mirror symmetry. This viewpoint has been clarified in much detail in the works of Gross-Siebert.

As we can see from the influence of mirror symmetry, string has had a strong affect on the development of mathematics.

In today's talk, I would like to tell you about a more recent developing area of string math collaboration. This is the study of non-Kähler manifolds with trivial canonical bundle. They are sometimes called non-Kähler Calabi-Yaus. For string theory, they play an important role as they appear in supersymmetric flux compactifications. But let me begin by describing to you why mathematicians were interested in them prior to string theory.

Non-Kähler Calabi-Yau

A large class of compact non-Kähler Calabi-Yau threefolds were already known in the mid-1980s by a construction of Clemens and Friedman. Their construction starts from a smooth Kähler Calabi-Yau threefold, Y.

Suppose Y contains a collection of mutually disjoint rational curves. These are curves that are isomorphic to $\mathbb{CP}^1$ and have normal bundles $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$. Following Clemens, we can contract these rational curves and obtain a *singular* Calabi-Yau threefold X_0 with ordinary double-point singularities. Friedman then gave a condition to deform X_0 into a smooth complex manifold X_t. What I have described is just the compact version of the local conifold transition which physicists are familiar with.

$$Y \dashrightarrow X_0 \dashrightarrow X_t$$

X_t's canonical bundle is also trivial. So it is a Calabi-Yau, but in general it is non-Kähler. To see this, we can certainly contract enough rational curves so that $H^2(Y)$ is killed and $b_2 = 0$. In this case, after smoothing, we end up with a complex non-Kähler complex manifold which is diffeomorphic to a k-connected sum of $S^3 \times S^3$, with $k \geq 2$.

In 1987, Reid put forth an interesting proposal (called Reid's fantasy). He wanted to make sense of the vast collection of diverse Calabi-Yau threefolds. He speculated that all (Kähler) Calabi-Yau threefolds, that can be deformed to Moishezon manifolds, fit into a single universal moduli space in which families of smooth Calabi-Yaus of different homotopy types are connected to one another by the Clemens-Friedman conifold transitions that I just described.

Now if we want to test this proposal, understanding non-Kähler Calabi-Yau manifolds becomes essential. For example, a question one can ask is what geometrical structures exist on these non-Kähler Calabi-Yau manifolds. If the metric is no longer Kähler, does it have some other property?

Balanced Metrics

A good geometric structure to consider is the one studied by Michelsohn in 1982. Recall that a hermitian metric, ω , is Kähler if

$$d\omega = 0 \qquad\qquad (Kahler) \; .$$

For threefolds, Michelsohn analyzed the weaker *balanced* condition,

$$d(\omega \wedge \omega) = 2\,\omega \wedge d\omega = 0 \qquad (balanced) \; .$$

Clearly, a Kähler metric is always balanced but a balanced metric may not be Kähler.

The balanced condition has also good mathematical properties. It is preserved under proper holomorphic submersions and also under birational transformations (Alessandrini-Bassanelli).

There are also simple non-Kähler compact balanced manifolds. For examples:

- Calabi showed that a complex tori bundle over a Riemann surface can not be Kähler, but it does have a balanced metric (Gray).
- The natural metric on a compact six-dimensional twistor spaces is balanced. As Hitchin showed, only those associated with S^4 and $\mathbb{CP}^2$ are Kähler. One can get a non-Kähler Calabi-Yau by taking branched covers of twistor spaces. Sometimes if the four-manifold is an orbifold, the singularities on the twistor space may be resolved to also give a non-Káhler Calabi-Yau.
- Moishezon spaces.

How about the non-Kähler Calabi-Yaus from conifold transitions?

With Jixiang Fu and Jun Li (2008), we rigorously proved:

Theorem (Fu-Li-Yau)

Let Y be a smooth Kähler Calabi-Yau threefold and let $Y \to X_0$ be a contraction of mutually disjoint rational curves. Suppose X_0 can be smoothed to a family of smooth complex manifolds X_t. Then for sufficiently small t, X_t admit smooth balanced metrics.

Our construction provides balanced metrics on a large class of threefolds. In particular,

Corollary

There exists a balanced metric on $\#_k(S^3 \times S^3)$ for any $k \geq 2$.

Knowing that a balanced metric is present is useful. But to really understand Reid's proposal for Calabi-Yau moduli space, it is important to define some *canonical* balanced metric which would satisfy an additional condition, like the Ricci-flatness condition for the Kähler Calabi-Yau case.

We would like to have a natural condition, and here string theory gives some suggestions. As Calabi-Yaus have played an important role in strings, one may ask what would be the natural setting to study compact conifold transitions and non-Kähler Calabi-Yau in physics.

Physicists have been interested in non-Kähler manifolds for more than a decade now in the context of compactifications with fluxes and model building. In this scenario, if one desire compact spaces without singularities from branes, then one should consider working in heterotic string.

For heterotic string, the conditions for preserving $N = 1$ supersymmetry with H-fluxes was written down by Strominger in 1986. Strominger's system of equations specifies the geometry of a complex threefold X (with a holomorphic three-form Ω) and in addition a holomorphic vector bundle E over X.

Strominger's System

The hermitian metric ω of the manifold X and the metric h of the bundle E satisfy the system of differential equations:

(1) $d(\| \Omega \|_\omega \, \omega \wedge \omega) = 0$

(2) $F_h^{2,0} = F_h^{0,2} = 0\,, \quad F_h \wedge \omega^2 = 0$

(3) $i\partial\overline{\partial}\,\omega = \frac{\alpha'}{4}\left[\mathrm{tr}\big(R_\omega \wedge R_\omega\big) - \mathrm{tr}\big(F_h \wedge F_h\big)\right]$

Notice that the first equation is equivalent to the existence of a (conformally) balanced metric. The second is the Hermitian-Yang-Mills equations which is equivalent to E being a stable bundle. The third equation is the anomaly equation.

When E is the tangent bundle T_X and X is Kähler, the system is solved with $h = \omega$, the Kähler Calabi-Yau metric.

Using a perturbation method, Jun Li and I have constructed smooth solutions on a class of Kähler Calabi-Yau manifolds with *irreducible* solutions for vector bundles with gauge group $SU(4)$ and $SU(5)$. Andreas and Garcia-Fernandez have generalized our construction on Kähler Calabi-Yau manifolds for any stable bundle E that satisfies $c_2(X) = c_2(E)$.

In recent years, my collaborators and I and other groups have also constructed solutions of the Strominger system on non-Kähler Calabi-Yaus.

As clear in heterotic string, understanding stable bundles on Calabi-Yau threefolds is important.

Donagi, Pantev, Bouchard and others have done nice work constructing stable bundles on Kähler Calabi-Yaus to obtain realistic heterotic models of nature.

Andreas and Curio have done analysis on the Chern classes of stable bundles on Calabi-Yau threefolds, verifying in a number of cases a proposal of Douglas-Reinbacher-Yau.

But returning to conifold transitions on compact Calabi-Yaus, I have proposed using Strominger's system to study Reid's proposal. Certainly the first condition that there is a balanced metric is not an issue. As I mentioned, Fu-Li-Yau have shown the existence of a balanced metric under conifold transitions. However, the heterotic string in the second condition adds a stable gauge bundle into the picture. So one needs to know about the stability of holomorphic bundle through a global conifold transition.

My student Ming-Tao Chuan in his recent PhD thesis examined how to carry a stable vector bundle through a conifold transition, from a Kähler to a non-Kähler Calabi-Yau. He makes one assumption that the initial stable holomorphic bundle is trivial in a neighborhood of the contracting rational curves. In this case, he proved that the resulting holomorphic bundle on the non-Kähler Calabi-Yau also has a hermitian Yang-Mills metric, and hence is stable.

So it is clear that two of the three conditions of Strominger's system - existence of a balanced metric and a hermitian-Yang-Mills metric on the bundle, can be satisfied. The last condition - the anomaly equation - which couples the two metric is perhaps the most demanding.

Jixiang Fu and I analyzed carefully the anomaly equation when the manifold is a T^2 bundle over a $K3$ surface. In this case, the anomaly equation reduces down to a Monge-Ampère type equation on $K3$:

$$\triangle(e^u - \frac{\alpha'}{2} f e^{-u}) + 4\alpha' \frac{\det u_{i\bar{j}}}{\det g_{i\bar{j}}} + \mu = 0,$$

where f and μ are functions on $K3$ satisfying $f \geq 0$ and $\int_{K3} \mu = 0$.

It would be interesting to show that the anomaly equation can be satisfied throughout the non-Kähler Calabi-Yau moduli space.

Symplectic Conifold Transitions: Smith-Thomas-Yau

So far, I have talked about conifold transitions between Calabi-Yau's that although can be non-Kähler, they nevertheless maintain a complex structure. The contraction of a rational curve, $\mathbb{CP}^1$ (and the inverse operation of resolution) is naturally a complex operation. The smoothing of a conifold singularity by S^3 on the other hand is naturally symplectic. Friedman's condition is needed to ensure that a smoothed out Calabi-Yau contains a global complex structure.

But instead of preserving the complex structure, we can preserve the symplectic structure throughout the conifold transition. This would be the symplectic *mirror* of the Clemens-Friedman's conifold transition. In this case, we would collapse disjoint Lagrangian three-spheres, and then replace them by symplectic two-spheres. Such a symplectic transition was proposed in a work of Ivan Smith, Richard Thomas and myself in 2002.

Locally, of course, there is a natural symplectic form in resolving the singularity by a two-sphere. But there may be obstructions to patching the local symplectic form to get a global one. Smith-Thomas-Yau wrote down the condition (analogous to Friedman's complex condition) that ensures a global symplectic structure. This symplectic structure however may not be compatible with the complex structure.

So in general, the symplectic conifold transitions result in non-Kähler manifolds, but they all have $c_1 = 0$ and so they are called symplectic Calabi-Yaus. In fact, Smith, Thomas, and I used conifold transitions to construct many real six-dimensional non-Kähler symplectic Calabi-Yaus.

In the symplectic conifold transition, if we can collapse all disjoint three-spheres, then such a process should result in a manifold diffeomorphic to a connected sums of $\mathbb{CP}^3$. This mirrors the complex case, which after collapsing all disjoint rational curves, gives a connected sums of $S^3 \times S^3$.

More recently, Fine-Panov have also constructed interesting simply-connected symplectic Calabi-Yaus with Betti number $b_3 = 0$, which means that they can not be Kähler.

As I mentioned, a balanced structure can always be found in a complex conifold transition. So similarly, we can ask if there is any geometric structure present before and after a symplectic conifold transition? Here we would be looking for a condition on the globally $(3,0)$-form which in the general non-Kähler case is no longer d-closed.

Again, we can turn to string theory for a suggestion. Is there a mirror dual of a complex balanced manifold in string that is symplectic and generally non-Kähler?

Such a symplectic mirror will not be found in heterotic string. All supersymmetric solutions satisfy the Strominger system in heterotic string. So the mirror dual of a complex balanced manifold with bundle should be another complex balanced manifold with bundle.

It turns out the answer can be found in type II string theories. As I will describe shortly, the equations for non-Kähler Calabi-Yau in type II string also give new insights into the natural cohomologies on non-Kähler manifolds.

Type II Strings: Non-Kähler Calabi-Yau Mirrors

In type II string theory, supersymmetric compactifications preserving a $SU(3)$ structure have been studied by many people in the last ten years. Since we are interested in non-Kähler geometries of compact manifolds, any supersymmetric solution will have orientifold sources. The type of sources help determine the type of non-Kähler manifolds. I will describe the supersymmetric equations written in a form very similar to that in Grana-Minasian-Petrini-Tomasiello (2005) and Tomasiello (2007). My description below is from joint work with Li-Sheng Tseng.

Complex Balanced Geometry in Type IIB

The supersymmetric equations that involve complex balanced threefolds is found in type IIB theory in the presence of orientifold 5-branes (and possibly also D5-branes). These branes are wrapped over holomorphic curves. In this case, the conditions on the hermitian $(1,1)$-form ω and $(3,0)$-form Ω can be written as

$$\begin{aligned} d\Omega &= 0 && (\textit{complex integrability}) \\ d(\omega \wedge \omega) &= 0 && (\textit{balanced}) \\ 2i\,\partial\bar{\partial}(e^f \omega) &= \rho_B && (\textit{source}) \end{aligned}$$

where ρ_B is the sum of Poincaré dual currents of the holomorphic curves that the five-brane sources wrap around, and f is a distribution that satisfies

$$i\,\Omega \wedge \overline{\Omega} = 8\, e^{-f}\, \omega^3/3!\ .$$

The balanced and source equations are interesting in that they look somewhat similar to the Maxwell equations. If one notes that $*\omega = \omega^2/2$ (where the $*$ is with respect to the compatible hermitian metric), then the equations can be expressed up to a conformal factor as

$$d(\omega^2/2) = 0$$

$$2i\,\partial\overline{\partial} * (\omega^2/2) = \rho_B$$

Now this is somewhat expected as the five-brane sources are associated with a three-form field strength F_3 which is hidden in the source equation. These two equations however do tell us something more.

Recall the Maxwell case. The equations in four-dimensions are

$$d\,F_2 = 0\,,$$

$$d * F_2 = \rho_e\,,$$

where ρ_e is the Poincaré dual current of some electric charge configuration.

Now, if we consider the deformation $F_2 \to F_2 + \delta F_2$ with the source fixed, that is $\delta\rho_e = 0$, this leads to

$$d(\delta F_2) = d * (\delta F_2) = 0\;,$$

which is the harmonic condition for a degree two form in de Rham cohomology. So clearly, the de Rham cohomology is naturally associated with Maxwell's equations.

For type IIB complex balanced equations, we can also deform $\omega^2 \to \omega^2 + \delta\omega^2$. Now if we impose that the source currents and the conformal factor remains fixed, then we have the conditions

$$d(\delta\omega^2) = \partial\bar{\partial} * (\delta\omega^2) = 0 \ ,$$

which turn out to be the harmonic condition for a (2,2)-element of the Bott-Chern cohomology:

$$H^{p,q}_{BC} = \frac{\ker d \cap \mathcal{A}^{p,q}}{\mathrm{im}\ \partial\bar{\partial} \cap \mathcal{A}^{p,q}} \ .$$

This cohomology was introduced by Bott-Chern and Aeppli in the mid-1960s.

The string equations thus points to the Bott-Chern cohomology as the natural one to use for studying complex balanced manifolds. Note when the manifold is Kähler, the $\partial\bar{\partial}$-lemma holds and the Bott-Chern and Dolbeault cohomology are in fact isomorphic.

Symplectic Mirror Dual Equations in Type IIA

The mirror dual to the complex balanced manifold is found in type IIA string. Roughly, the type IIA equations can be obtained from the IIB equations, by first replacing $\omega^2/2$ with $(\mathrm{Re}\, e^{i\,\omega})$ and then exchanging $e^{i\,\omega}$ with Ω.

$$d(\omega^2/2) = 0 \quad \Leftrightarrow \quad d(\mathrm{Re}\, e^{i\,\omega}) = 0 \qquad \longleftrightarrow \qquad d\,\mathrm{Re}\;\Omega = 0$$

Thus, $d\,\mathrm{Re}\,\Omega = 0$ is the condition that is suggested by string for symplectic conifold transition.

This condition is part of the type IIA supersymmetric conditions in the presence of orientifold (and D-) six-branes wrapping special Lagrangian submanifolds:

The type IIA equations that are mirrored to the IIB complex balanced system are

$$
\begin{aligned}
d\omega &= 0\,, && (\textit{symplectic})\\
d\,\mathrm{Re}\;\Omega &= 0\,, && (\textit{almost complex})\\
\partial_+\partial_- * (e^{-f}\,\mathrm{Re}\,\Omega) &= \rho_A\,, && (\textit{source})
\end{aligned}
$$

where ρ_A is the Poincaré dual of the wrapped special Lagrangian submanifolds. ∂_+ and ∂_- are linear symplectic operators that can be thought of as the symplectic analogues of the Dolbeault operators, ∂ and $\overline{\partial}$. Tseng and I introduced them recently, so let me describe them a little bit more.

(∂_+, ∂_-) appear from a symplectic decomposition of the exterior derivative

$$d = \partial_+ + \omega \wedge \partial_- \ .$$

∂_+ raises the degree of a differential form by one, and ∂_- lowers the degree by one. They are defined with the property

$$\partial_\pm : \mathcal{P}^k \to \mathcal{P}^{k\pm 1} \ ,$$

where $\mathcal{P}^k$ is the space of primitive k-form. (A primitive form is one that vanishes after being contracted with ω^{-1}.) And like their complex counterparts,

$$(\partial_+)^2 = (\partial_-)^2 = 0 \ ,$$

and effectively, they also anticommute with each other.

With the linear symplectic operators, (∂_+, ∂_-), we can write down an interesting elliptic complex.

Proposition (Tseng-Yau)

On a symplectic manifold of dimension $d = 2n$, the following differential complex is elliptic.

$$\begin{array}{ccccccccccc}
0 & \xrightarrow{\partial_+} & \mathcal{P}^0 & \xrightarrow{\partial_+} & \mathcal{P}^1 & \xrightarrow{\partial_+} & \dots & \xrightarrow{\partial_+} & \mathcal{P}^{n-1} & \xrightarrow{\partial_+} & \mathcal{P}^n \\
 & & & & & & & & & & \Big\downarrow{\scriptstyle \partial_+\partial_-} \\
0 & \xleftarrow{\partial_-} & \mathcal{P}^0 & \xleftarrow{\partial_-} & \mathcal{P}^1 & \xleftarrow{\partial_-} & \dots & \xleftarrow{\partial_-} & \mathcal{P}^{n-1} & \xleftarrow{\partial_-} & \mathcal{P}^n
\end{array}$$

Associated with this elliptic complex are four different finite-dimensional cohomologies which gives new symplectic invariants for non-Kähler manifolds.

Symplectic cohomologies (Tseng-Yau):

Symplectic (X,ω)	Complex (X,J)
$PH^s_{\partial_\pm} = \dfrac{\ker \partial_\pm \cap \mathcal{P}^s}{\mathrm{im}\ \partial_\pm \cap \mathcal{P}^s}$	$\dfrac{\ker \bar\partial \cap \mathcal{A}^{p,q}}{\mathrm{im}\ \bar\partial \cap \mathcal{A}^{p,q}}$ (Dolbeault)
$PH^s_{\partial_+\partial_-} = \dfrac{\ker \partial_+\partial_- \cap \mathcal{P}^s}{(\mathrm{im}\ \partial_+ + \mathrm{im}\ \partial_-) \cap \mathcal{P}^s}$	$\dfrac{\ker \partial\bar\partial \cap \mathcal{A}^{p,q}}{(\mathrm{im}\ \partial + \mathrm{im}\ \bar\partial) \cap \mathcal{A}^{p,q}}$ (Aeppli)
$PH^s_{\partial_++\partial_-} = \dfrac{\ker d \cap \mathcal{P}^s}{\mathrm{im}\ \partial_+\partial_- \cap \mathcal{P}^s}$	$\dfrac{\ker d \cap \mathcal{A}^{p,q}}{\mathrm{im}\ \partial\bar\partial \cap \mathcal{A}^{p,q}}$ (Bott-Chern, Aeppli)

The middle-degree cohomology

$$PH^n_{\partial_++\partial_-} = \frac{\ker d \cap \mathcal{P}^n}{\mathrm{im}\ \partial_+\partial_- \cap \mathcal{P}^n}$$

turns out to appear in type IIA string.

For consider the deformation: $\Omega \longrightarrow \mathrm{Re}\,\Omega + \delta\mathrm{Re}\,\Omega$ with $\delta\rho_A = 0$ and conformal factor remaining invariant. Then the $\delta\mathrm{Re}\,\Omega$ satisfy

$$d(\delta\mathrm{Re}\,\Omega) = 0\ , \qquad\qquad \partial_+\partial_- * (\delta\mathrm{Re}\,\Omega) = 0\ ,$$

which is the harmonic condition of the primitive $PH^n_{\partial_+ + \partial_-}$ cohomology.

In fact, a subspace of the linearized deformation of the type IIA symplectic system can be parametrized by the cohomology

$$\delta\Omega \in P\mathcal{H}^3_{\partial_+ + \partial_-} \cap \mathcal{A}^{2,1}\ .$$

Non-Kähler geometry on six-dimensional manifolds will have a lot activities in the near future. These geometries can have relations with four- and three-dimensional manifolds. One can construct non-Kähler six-manifolds by the twistor construction. The twistor space of an anti-self dual four-manifolds has a complex structure, and the twistor space of a hyperbolic four-manifold has a symplectic structure. The S^3 bundle over a hyperbolic three-manifold is also complex. (Fine-Panov have given examples of the hyperbolic constructions.) There should also be interesting dualities relating complex and symplectic structures on non-Kähler six-manifolds.

The major guiding influence will be string theory.