Orbifold and Local Heterotic Flux Geometry

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Heterotic Models

** Compactifications: phenomenologically interesting, Natural gauge group and Standard Model fields (Works of Penn Math/Physics Group and many speakers in this conference.)

** Add Fluxes and Branes, geometry backreacts and becomes no longer Calabi-Yau.

** Lift scalar moduli [See M. Becker talk]

Outline

I. Review heterotic SUSY constraint & mathematical motivation
Background and References [See M. Becker talk]

II. Orbifold solutions
Geometric quotient of T^2 bundle over $K3$ solution.

III. Local non-compact solutions
A heterotic model on ALE space: Eguchi-Hanson space $T^*\mathbb{P}^1$.

Works with M. Becker and J.X. Fu, to appear.

I. Heterotic SUGRA $N = 1$ SUSY Costraints

Physical fields: (g, H_3, ϕ, F_2)
Geometry: (X_6, E) $[M^{3,1} \times X_6$ & gauge bundle$]$

1. $SU(3)$ structure (J, Ω)

   $J = J_{a\bar{b}}\, dz^a \wedge d\bar{z}^{\bar{b}} = i g_{a\bar{b}}\, dz^a \wedge d\bar{z}^{\bar{b}}$
   $\Omega^{3,0}$ $\leftarrow$ defines an almost complex structure

   $i\,\Omega \wedge \bar{\Omega} = \frac{4}{3}\|\Omega\|^2 J \wedge J \wedge J \qquad \|\Omega\| = e^{-2\phi}$

2. Complex

   $$d\Omega = 0$$

   Hence, class $c_1(X_6) = 0$ holomorphically.

3. Balanced Metric (conformal)

$$d(\|\Omega\| * J) = 0$$

4. Hermitian Yang-Mills (equivalent to the condition that E is a "stable" bundle, Li-Yau '86)

$$F^{2,0} = F^{0,2} = 0\,, \qquad F \wedge J \wedge J = 0 \Leftrightarrow F_{mn}J^{mn} = 0$$

5. Anomaly Equation ($H = i(\bar\partial - \partial)J$)

$$2i\,\partial\bar\partial J = \frac{\alpha'}{4}\,(\mathrm{tr} R \wedge R - \mathrm{tr} F \wedge F)$$

Balanced Manifold (Michelsohn '82)

Kähler: $dJ = 0$

Balanced: $d(*J) = \frac{1}{2}d(J \wedge J) = J \wedge dJ = 0$

** weaker, relaxation of Kähler condition

** Preserved under smooth blowing down in 6D (Alessandrini-Bassanelli '95).

** Preserved under conifold transitions (Fu-Li-Yau '08). Important for the Reid conjecture connecting CY_3.

Examples: Iwasawa manifolds (T^2 bundle over T^4), twistor spaces, connected sums of $(S^3 \times S^3)$.

Anomaly Condition

$C_1(X_6) = 0$

Ricci-flat: $c_1(X_6) = \frac{i}{2\pi}\mathrm{tr}R_{a\bar{b}} = \lambda J$ with $\lambda = 0$.

Anomaly: $p_1(X_6) = \frac{1}{8\pi^2}\mathrm{tr}R \wedge R = \frac{1}{8\pi^2}\mathrm{tr}F \wedge F + \frac{i}{4\pi^2}\partial\bar{\partial}J$

** Cohomlogy class level, $[c_1(X_6)] = 0$ and
and $[p_1(X_6)] = [p_1(E)]$.

** Involves analysis of 4-form characteristic invariants

** Highly non-linear

Summary: Heterotic $N = 1$ Constraints (Strominger system)

Let X_6 be a hermitian manifold with a stable gauge bundle E. Topologically, we require

1. $C_1(X_6) = 0$
2. $P_1(X_6) = P_1(E)$
3. $\exists$ a positive $(2, 2)$ form on X_6

Solve, for (J, Ω) on X_6 and F the curvature on E

A. $d\Omega = 0$
B. $d(\|\Omega\| * J) = 0$
C. $F^{2,0} = F^{0,2} = 0\,, \quad F \wedge J \wedge J = 0$
D. $2i\,\partial\bar{\partial} J = \frac{\alpha'}{4}(\mathrm{tr} R \wedge R - \mathrm{tr} F \wedge F)$

II. Orbifolds of T^2 bundle over $K3$ (FSY) solutions

We start with the FSY solution

Geometry:

$$\begin{array}{ccc} T^2 & \longrightarrow & X_6 \\ & & \downarrow \\ & & K3 \end{array} \qquad\qquad \begin{array}{ccc} E_X & \longrightarrow & X_6 \\ \downarrow & & \downarrow \\ E_{K3} & \longrightarrow & K3 \end{array}$$

Gauge bundle: E_X stable bundle on X_6 lifted from $K3$

(J, Ω) : $\qquad J = e^u J_{K3} + \frac{i}{2}\theta \wedge \bar{\theta} \qquad \Omega = \Omega_{K3} \wedge \theta$

where $\qquad \theta = dz + \alpha = (dx + \alpha_1) + \tau(dy + \alpha_2)$

is defined to be a $(1,0)$-form.

$\omega = d\theta = \omega_1 + \tau\omega_2 \,, \qquad \|\Omega\|^2 = e^{-2u} = e^{-4\phi}$

(ω, F, u) fixed by the following requirements:

1. Complex: $\omega = d\theta \in H^{2,0}(K3, \mathbb{Z}) \oplus H^{1,1}(K3, \mathbb{Z})$

2. Conformally balanced: $\omega \wedge J_{K3} = 0$

3. F: the hermitian Yang-Mills curvature associated with the stable bundle on E_X.

4. Anomaly condition: $\int_{K3} \omega \wedge \bar{\omega} + \int_{K3} \mathrm{tr} F \wedge F = \int_{K3} \mathrm{tr} R \wedge R = 24$

Fu-Yau showed that there exists a solution to the non-linear system of differential equations.

Constructing new solutions by orbifolding FSY

For T^2 bundle over $K3$ geometries with a discrete symmetry, we construct new solutions by quotienting the geometry, X_6/Γ, where Γ is a finite group action.

Require the discrete symmetry to leave invariant the physical fields

$$g_{mn} = J_m{}^r J_{rn}$$

$$H = d^c J$$

$$e^{-4\phi} = \|\Omega\|^2$$

This is satisfied as long as J is invariant and $\Omega \to \zeta\Omega$ where $|\zeta| = 1$. If $\zeta \neq 1$, then the resulting orbifold solution breaks all supersymmetry.

Discrete symmetry action can have two components, one acting on the fiber and the other on the base. Let N be the order of the finite group. Separately, we have

Fiber T^2

(1) shift $\rho : z \to z + c$ $Nc = a + b\tau$

(2) rotation $\rho : z \to \zeta z$ $\zeta^N = 1$

Base $K3$:

(1) symplectic $\rho : \Omega^{2,0} \to \Omega$

(2) non-symplectic $\rho : \Omega^{2,0} \to \zeta\Omega$ $\zeta^N = 1$

Must be algebraic $K3$ surfaces

Classification: Nikulin ($\mathbb{Z}_2$); Artebani & Sarti, Taki ($\mathbb{Z}_3$)

Construct Solutions:

(1) Start with $K3$ surfaces with discrete symmetry ρ
(2) The curvature twist ω_1, ω_2 of T^2 sits in the lattice L of $H^2(K3, \mathbb{Z})$ such that
(a) choose primitive ω_1, ω_2 that transforms similarly to the action on the torus action such that
(b) $\omega = \omega_1 + \tau\omega_2 \in H^{2,0}(K3, \mathbb{Z}) \oplus H^{1,1}(K3, \mathbb{Z})$
(c) $\int_{K3} \omega \wedge \bar{\omega} = 24$.

Construct examples below. First consider torus action

1. Shift: $z \to z + c$
**no fixed points, always smooth
**SUSY N=2,1,0
**Reduce size of the torus fiber along fixed points on the base

Example: $K3$ as a triple cover of $\mathbb{P}^1 \times \mathbb{P}^1$ branched over a curve

$K3$ as the interesection of two hypersurfaces.
$\mathbb{P}^4 : [z_0, z_1, z_2, z_3, z_4]$

$f^1 = z_0 z_3 - z_1 z_2$ (embed $\mathbb{P}^1 \times \mathbb{P}^1$ in $\mathbb{P}^3$)
$f^2 = g_3(z_0, z_1, z_2, z_3) + z_4^3$

$\mathbb{Z}_3$ action: $\rho(z_0, z_1, z_2, z_3, z_4) = (z_0, z_1, z_2, z_3, \zeta z_4)$
One fixed genus 4 curve at $g_3(z_0, z_1, z_2, z_3) = 0$

$\rho : \Omega^{2,0} \to \zeta \Omega^{2,0}$ with $\zeta^3 = 1$.
$\omega \sim \omega_A - \omega_B$ invariant

2. Rotations: $z \to \zeta z$

**Since $\theta = dz + \alpha$ is a global 1-form, action must be non-trivial on the base.
** Fixed locus set is non-empty.
***Generically, must resolve singularities (points and curves) to get a smooth manifold.

For triple cover K3, can choose ω that transforms non-trivially and obtain a SUSY solution

$$\begin{array}{ccc} T^2 & \longrightarrow & X_6' \\ & & \downarrow \\ & & \mathbb{P}^1 \times \mathbb{P}^1 \end{array}$$

but must resolve the singularities along the branched curve.

Example: $K3$ surface with $\mathbb{Z}_3$ action with only fixed points

As before, intersection of a degree 2 and a degree 3 hypersurface in $\mathbb{P}^4 : [z_0, z_1, z_2, z_3, z_4]$.

$$f^1 = f_2(z_0, z_1) + b_1 z_2 z_3 + b_2 z_2 z_4$$
$$f^2 = f_3(z_0, z_1) + b_3 z_2^3 + g_3(z_3, z_4) + z_2 f_1(z_0, z_1) g_1(z_3, z_4)$$

For example, $f^1 = z_0^2 + z_1^2 + z_2(z_3 + z_4) = 0$
$$f^2 = z_1^3 + z_2^3 + z_3^3 - z_4^3 = 0$$

$\mathbb{Z}_3$ action: $\rho(z_0, z_1, z_2, z_3, z_4) = (\zeta^2 z_0, \zeta^2 z_1, \zeta z_2, z_3, \zeta z_4)$
3 fixed points at $(z_0, z_1, z_2) = (0, 0, 0)$ and $g_3(z_3, z_4) = 0$

$\rho(\Omega^{2,0}) = \zeta^2\Omega$ hence, $\rho(\theta) = \zeta\theta$ if we want to preserve SUSY $\Omega = \Omega^{2,0} \wedge \theta$.

Take $\tau = e^{2\pi i/3}$.

ω_1, ω_2 are in the $N_\rho^\perp = U(1) \oplus U(3) \oplus A_2^5 \subset L$,
chosen such that $\omega = \omega_1 + \tau\omega_2 \in H^{1,1}(K3, \mathbb{Z})$
i.e. orthognal to $\Omega^{2,0}$ and $\bar{\Omega}^{0,2}$.

Resolution: Blow up fixed points with boundary $\mathbb{C}^3/\mathbb{Z}_3$.

III. Local Model with Eguchi-Hanson base

Metric ansatz:

$$J = e^u J_{CY_2} + \frac{i}{2}\,\theta \wedge \bar{\theta}$$

Take the base CY_2 to be an ALE space. Simplest is the Eguchi-Hanson space: blow up of $\mathbb{C}^2/\mathbb{Z}_2$ at the origin of the $\mathbb{Z}_2$ action $\sigma(z_1, z_2) = (-z_1, -z_2)$. Alternatively, $B = \mathcal{O}^1_{\mathbb{P}}(-2) = T^*\mathbb{P}^1$. There is a Ricci-flat metric

$$J_{EH} = \frac{i}{2}(k(r^2)\partial\bar{\partial}r^2 + k'(r^2)\partial r^2 \wedge \bar{\partial}r^2)$$

$k = \sqrt{1 + \frac{a^4}{r^4}}$ and $r^2 = |z_1|^2 + |z_2|^2$ radius on $\mathbb{C}^2$.
a is the size of the blow-up $\mathbb{P}^1$.

On EH, there is a single anti-self dual (1,1)-form. We can use this to twist the torus and as $U(1)$ gauge fields.

$$\omega \sim i(h(r^2)\partial\bar{\partial}r^2 + h'(r^2)\partial r^2 \wedge \bar{\partial}r^2)$$

where $h(r^2) = \frac{1}{a^2 r^2 \sqrt{1+\frac{r^4}{a^4}}}$.

We need to satisfy the anomaly equation. Much simplification due to dependence only on the radial coordinate for all quantities on $\mathbb{C}^2/\mathbb{Z}_2$. The differential equation can be written as

$$\begin{aligned} 0 &= dH - \frac{\alpha'}{4}\left[\mathrm{tr}R \wedge R - \mathrm{tr}F \wedge F\right] \\ &= \frac{\left[A(r^2)\, r^4\right]'}{r^2} dz_1 \wedge d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_2 \end{aligned}$$

where

$$A(r^2) = -u'e^u \frac{a^2}{r^2}\sqrt{1+\frac{r^4}{a^4}} + \alpha' \frac{(|n|^2 + \frac{n_i^2}{2})}{r^4(1+\frac{r^4}{a^4})}$$

$$-\alpha' \left[\frac{3}{r^4(1+\frac{r^4}{a^4})^2} + (u')^2 + \alpha'|n|^2 e^{-u} \left(\frac{u'}{a^2 r^2 (1+\frac{r^4}{a^4})^{3/2}} + \frac{4}{a^6(1+\frac{r^4}{a^4})^{5/2}} \right) \right]$$

$$= 0$$

where $|n|^2 = n_1^2 + n_2^2$, n_1, n_2, and n_i' are the first Chern number of the torus bundle and $U(1)$ gauge bundle.

We find a smooth solution for $|n|^2 + \frac{n_i^2}{2} = 3$, which corresponds to matching characteristic classes on the EH base.

Convergent solution for α'/a^2 sufficiently small.

$$e^u = \sum_{k=0}^{\infty} \frac{a_k}{(1+\frac{r^4}{a^4})^{\frac{k}{2}}}$$

$$= 1 - \frac{\alpha'}{a^2}\frac{1}{(1+\frac{r^4}{a^4})^{\frac{3}{2}}} + \left(\frac{\alpha'}{a^2}\right)^2 \frac{|n|^2}{(1+\frac{r^4}{a^4})^2} + \left(\frac{\alpha'}{a^2}\right)^3 \frac{(|n|^2+9/7)}{(1+\frac{r^4}{a^4})^{\frac{7}{2}}} + \ldots$$

Physical Implications

**Solution has non-zero fractional H_3 charge, sourced by the twist of the T^2 and gauge fields.

**Five-brane charge is generated when wrapped on twisted T^2 bundle.

**Expect higher order in α' corrections of the differential equation and solution.

In Summary

** The study of heterotic torsional solutions are phenomenologically important and provides a good framework for investigating new mathematics.

** The space and structure of solutions is currently not well-understood, except for specific cases.

** Expect new exciting results in the future.