New Calabi-Yau 3-folds and their mirrors via conifold transitions

Maximilian Kreuzer / Vienna University of Technology

Work with

- V. **Batyrev** (Tübingen): **...** **conifold transitions** //arXiv:0802.3376 [math.AG]
- –〃–: **Integral cohomology** & **mirror symmetry** //arXiv:math.AG/0505432
- V. **Braun**, B.A. **Ovrut** (U.Penn) and E. **Scheidegger** (Augsburg)
  Worldsheet instantons, torsion curves, non-perturbative superpotentials
  arXiv:hep-th/0703134, hep-th/0703182, 0704.0449 [hep-th]
- A. **Klemm** (Bonn), E. **Riegler** (Vienna) and E. **Scheidegger** (Augsburg)
  Topological strings, CICYs & threshold corrections //arXiv:hep-th/0410018

Supported by the Austrian Research Fund FWF

Content

- Motivation to study the web of Calabi-Yau manifolds
- Geometry and Combinatorics: Reflexive Polytopes
- Torsion in Cohomology: Fundamental $\overset{\text{MS}}{\longleftrightarrow}$ Brauer group
- Beyond hypersurfaces: Complete Intersections $\begin{cases} \text{more torsion in } H^* \\ \text{more general singularities} \end{cases}$

  $\subseteq$ Reflexive Cones: $\leftrightarrow$ classification rigid CYs

- Beyond toric CYs: Mirror Pairs via Conifold Transitions
  - Results: surprisingly many new CYs with small h_{11}

    1-parameter case: topologies & PF opertors
  - Construction: combinatorial $\longrightarrow$ 30241 cases with $h_{11} \approx 4$
- ToDo, ToFindOut & ToApply

Why should we classify Calabi-Yau 3-folds?

- Math: important part of the classificaiton of 3-folds
- Strings: Model building
  - Heterotic: fundamental groups, torsion // symmetric CYs (?)
  - Orientifolds: exceptional divisors // more generic singularities (?)
  - F-theory: elliptic 4-folds // $\exists$ too many: bottom up classification (?)
- What do we know?
  - only examples (finiteness ?); except: elliptic case [M. Gross]
  - but Reid's phantasie: connected by singular transitions
- Toric hypersurfaces: + very numerous (largest known list)
  + very conveneint (combinatorics, MS)
  – very special !!!
  + connected $\Rightarrow$ use as backbone of the web

Conifold transitions

- Candelas, Green, Hubsch '90: Other worlds are just around the corner
- Greene, Morrison, Strominger '95: Black hole condensation /N=2 SUSY
- Danielsson '07: ... landscape topography / N=1
- w/Batyrev [0802.3376]:
  - mirror to Candelas et al., blow down $\mathbb{P}^1 \longrightarrow$ flat deformation toric hypersurface $\longrightarrow$ reduce h_{11}
  - generalizing:Batyrev, Ciocan-Fontanine, Kim, van Straten: Mirror symmetry for complete intersections in Grassmannians & *Flag manifolds* via toric degenerations
- Combinatorial conditions: • 2-faces are minimal triangles or squares
  • smoothing condition: rank of matrix of linear relations

473 800 776 reflexive polyhedra: 4.5GB ... 1 day on desktop / 2-3 days on laptop

30241 new examples of (presumed) mirror pairs

Geometry and combinatorics: hep-th/0612307

- **Quintic** in $\mathbb{P}^4$: $z_0^5 + \ldots + z_4^5 = 0$ **affine coord.** $t_i = z_i/z_0 \in \mathbb{R}^n \subseteq \mathbb{P}^n$

  homogeneous polynomial $p(z) \Rightarrow f(t_j) = p(z_j)/z_i^d$ on patch $U_i \cong \mathbb{R}^n$

- $\mathbb{C}^*$ scaling: line bundle $\mathcal{O}(d)$: $g_{ij} = (\frac{z_i}{z_j})^d$ ... monomial transition function
  - quasi-homogeneous: weighted projective $W\mathbb{P}^n$
  - multi-quasi-homogeneous: toric variety $\mathbb{P}_\Sigma$ (for simplicial fan)

- $f(t_i) = \sum_{\vec{m} \in \Delta \cap M} c_m t^{\vec{m}}$ Newton polytope $\Delta \in M_\mathbb{R}$ of **exponent vectors** $\vec{m} \in M \cong \mathbb{Z}^n$

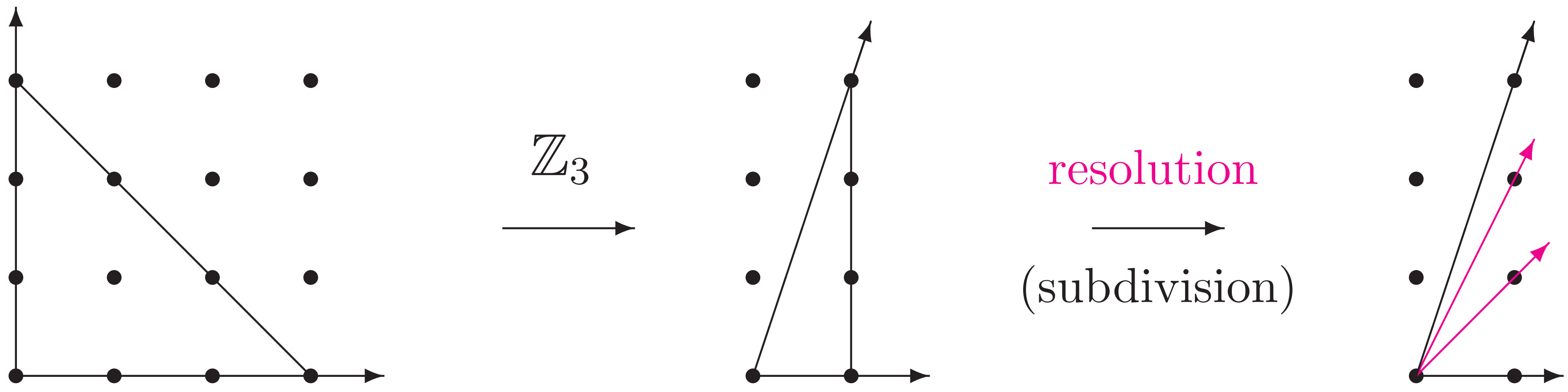
- affine coordinates $t_i = \prod_j z_j^{v_{ij}}$, $v_{ij} = \langle e_i, v_j \rangle \ldots v_j \in N = M^*$

- Monomials: $\xi_m = \prod_i t_i^{m_i} = \prod_j z_j^{\langle m, v_j \rangle}$ polytope $\nabla = \langle v_j \rangle \subseteq N_\mathbb{R}$

  fan Σ of **cones over** ∇

- $\{f = 0\}$ is Calabi-Yau $\Leftrightarrow \langle m, v_j \rangle \geq -1 \ \ldots \ \nabla = \Delta^*$ reflexive

Singularities: the $\mathbb{Z}_n$ quotient $\mathbb{C}[X,Y]\,/\,\mathbb{Z}_n : \begin{smallmatrix} X \to e^{2\pi i/n}\, X \\ Y \to e^{2\pi i/n}\, Y \end{smallmatrix}$



Invariant X^3, X^2Y, XY^2, Y^3 $\quad$ $\tilde{X} = X^3,$ $\quad$ $\tilde{X} = X_1,\ Y_1 = X_2,\ \ldots\ Y_3 = \tilde{Y}$

$X^6, X^5Y, X^4Y^2, \ldots$ $\quad$ $\tilde{X}\tilde{Y} = X^2Y,$ $\quad$ transition: $Y_2 = Y_1/X_1, \ldots$

cheating: subdivision of cone $\sigma \subset N_{\mathbb{R}}$ with $\sigma \in \Sigma$ $\quad$ new coordinates z_j

dual to the cone $\sigma^\vee \subset M_{\mathbb{R}}$ of monomials $X, Y, \ldots$

conifold: $xy = zw \ldots \sigma^\vee \in M_{\mathbb{R}}$ vs. triangulation of $\sigma \in N_{\mathbb{R}}$, cf. hep-th/0612307

Theorem:

- $\mathbb{P}_\Sigma$ is smooth iff all cones are simplicial and unimodular
- $Vol(\theta_k) > 1$ for k-face $\theta_k \in \Sigma \iff$ singularity of dimension $n-k-1$

because: # of vanishing homogeneous coordinates $\Rightarrow$ e.g. facet $\leftrightarrow$ point

Toric (hypersurface) dictionary:

• hypersurface $f_\Delta = 0$	$\Leftrightarrow$	Δ is reflexive
• (toric) fibration	$\Leftrightarrow$	$\exists$ reflexive section of $\nabla \subset N_\mathbb{R}$
• divisors	$\Leftrightarrow$	$v_j \in \Sigma^{(1)}$
• Hodge numbers	$\Leftrightarrow$	lattice points on θ_k
• fundamental group	$\Leftrightarrow$	index of sublattices $\langle \text{pt's} \rangle_\mathbb{Z}$
• ...	$\Leftrightarrow$	...

Torsion in (co)homology w/V. Batyrev [math.AG/0505432]

- Universal coefficient theorem

$$\mathrm{tor}(H_i(X,\mathbb{Z})) \cong \mathrm{tor}(H^{i+1}(X,\mathbb{Z}))^*$$

- Poincaré duality: $\mathrm{tor}(H_i(X,\mathbb{Z})) \cong \mathrm{tor}(H^{2d-i}(X,\mathbb{Z}))$
- 3-folds $\Rightarrow$ two independent torsion groups:

  $\mathrm{tor}\, H_1(X,\mathbb{Z}) \cong \mathrm{tor}\, H^2(X,\mathbb{Z})^*$ (related to fundamental group)

  $\mathrm{tor}\, H_2(X,\mathbb{Z}) \cong \mathrm{tor}\, H^3(X,\mathbb{Z})^*$ (topological Brauer group)

Conjecture: The torsion subgroups of H^2 and H^3 are exchanged under the mirror involution

- verified for all 473 800 776 toric Calabi–Yau hypersurfaces: $16+16$ cases:

  $\mathbb{P}^4[5]/\mathbb{Z}_5$, elliptically fibered: $\mathbb{P}^2\times\mathbb{P}^2[3,3]/\mathbb{Z}_3$ and $\mathbb{P}_{11133}[9]/\mathbb{Z}_3$,

  $13\times$ $\pi_1(X)=\mathbb{Z}_2$ (elliptic-K3 fibered)

- fundamental group $\leftrightarrow$ index of sublattice $\langle\text{codim-2 points}\rangle_{\mathbb{Z}}$

  (Brauer group)^2 $\leftrightarrow$ index of sublattice $\langle\text{codim-3 points}\rangle_{\mathbb{Z}}$

  i.e. points on edges of ∇

Torsion curves for the "Heterotic standard model"

with V. Braun, B. Ovrut and E. Scheidegger

- Schoen CY: (3,3) parameter, fiber product of two elliptic fibers over $\mathbb{P}^1$
- complete intersection: $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^1 \begin{bmatrix} 3 & 0 & 1 \\ 0 & 3 & 1 \end{bmatrix} \Rightarrow$ Batyrev-Borisov mirror
- free $\mathbb{Z}_3$ phase (toric) $\times$ $\mathbb{Z}_3$ permutation (non-toric) group action
- Self-mirror! $\Rightarrow$ $\mathbb{Z}_3 \times \mathbb{Z}_3$ torsion curves (spectral sequence computation)
- Application: torsion curves cannot be holomorphic vs. Beasley–Witten no-go single curve in homology class! $\rightarrow$ SUSY breaking & moduli stabilization

Lessons for CYs:

- need codimension > 1 for having both fundamental + Brauer group
- for large torsion in H^* Candelas et al's '88 list of CICYs in products of $\mathbb{P}^n$'s is a good starting point (don't need complicated polytopes)

Apropos complete intersections

- $f_1 = \ldots = f_r = 0 \;\Rightarrow\;$ Minkowski sum $\Delta = \Delta_1 + \ldots + \Delta_r$
- Batyrev-Borisov mirror duality: $\nabla = \nabla_1 + \ldots + \nabla_r$ where

$$\Delta = \Delta_1 + \ldots + \Delta_r \qquad \Delta^* = \langle \nabla_1, \ldots, \nabla_r \rangle \mathrm{conv}$$

$$\langle \Delta_i, \nabla_j \rangle \geq -\delta_{ij}$$

$$\nabla^* = \langle \Delta_1, \ldots, \Delta_r \rangle \mathrm{conv} \qquad \nabla = \nabla_1 + \ldots + \nabla_r$$

Cayley trick: CICY $f_i = 0 \overset{\textit{complement}}{\longleftrightarrow}$ hypersurface $\sum t_i f_i = 0$

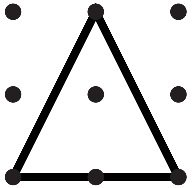
Reflexive Gorenstein cones of dimension $n+r$ for CY_{n-r}

$$\tilde{M}_{\mathbb{R}} \supseteq \tilde{\Delta} = \mathrm{Conv}(e_i, \Delta_i) \quad \overset{\textit{dual}}{\longleftrightarrow} \quad \tilde{\nabla} = \mathrm{Conv}(e_j, \nabla_j) \subseteq \tilde{N}_{\mathbb{R}} \cong \mathbb{R}^{n+r}$$

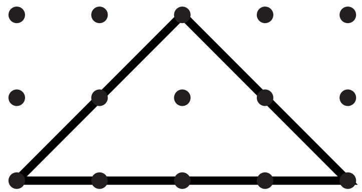
Batyrev, Nill: *Combinatorial aspects of mirror symmetry* [math/0703456]:

- Split Gorenstein cone $\Rightarrow$ CY $n-r$ fold (codimension r)
- Rigid CY: only one cone is split, the mirror is "generalized" CY in the sense of 1990s
- $\exists$ classification algorithm (work in progress)

Nef partitions & Batyrev–Borisov duality



$\Delta^\circ \in N \quad \to$ coordinates z_i sections $\sim \sum_{m\in\Delta} \prod_i z_i^{\langle m, v_i\rangle}$



$\Delta \in M = N^* \quad \to \quad$ line bundles $\leftrightarrow$ equations

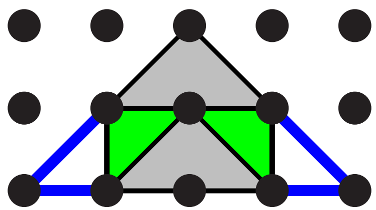
$\Rightarrow$ CICY: decompose $\Delta = \Delta_1 + \Delta_2$ (Minkowski sum)

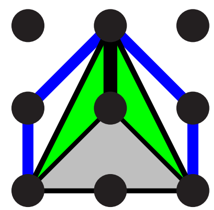
V.V. Batyrev & L.A. Borisov [alg-geom/9412017]:

– NEF partitions: piecewise linear convex "support functions" $\varphi_j(e_i) = \delta_{ij}$
numerically effective $\to$ ample line bundles

– combinatorial duality $\leftrightarrow$ mirror symmetry ... **4 reflexive polytopes:**

$$\begin{aligned} \Delta &= \Delta_1 + \Delta_2 \\ \nabla^* &= \langle \Delta_1, \Delta_2 \rangle \end{aligned} \qquad \langle \Delta_i, \nabla_j \rangle = \begin{cases} \geq -1 & \text{if } i = j \\ \geq 0 & \text{if } i \neq j \end{cases} \qquad \begin{aligned} \Delta^* &= \langle \nabla_1, \nabla_2 \rangle \\ \nabla &= \nabla_1 + \nabla_2 \end{aligned}$$





Mirror symmetry: duality extends to Hodge data

V.V.Batyrev, L.A.Borisov: alg-geom/9509009

$$\sum(-1)^{p+q}\, h_{pq}\, t^p\, \bar{t}^q = \sum_{I=[x,y]} \frac{(-)^{\rho_x} t^{\rho_y}}{(t\bar{t})^r} S(C_x, \frac{\bar{t}}{t}) S(C_y^*, t\bar{t}) B(I; t^{-1}, \bar{t})$$

– $C_x, C_y \in$ face lattice of Gorenstein cone spanned by (e_i, Δ_i)

– $B(I)$ encodes combinatorics of the sublattice $I = [x, y]$ with $x < y$

– $S(C_x, t) = (1-t)^{\rho_x} \sum_{m \in C_x} t^{\deg(m)}$ related to the Erhart polynomial

nef.x ($\in$ PALP) by Erwin Riegler [math.AG/0103214, math.CS/0204356]:

Batyrev's formula for codimension $r = 1$: codim 1 - divisors do not intersect

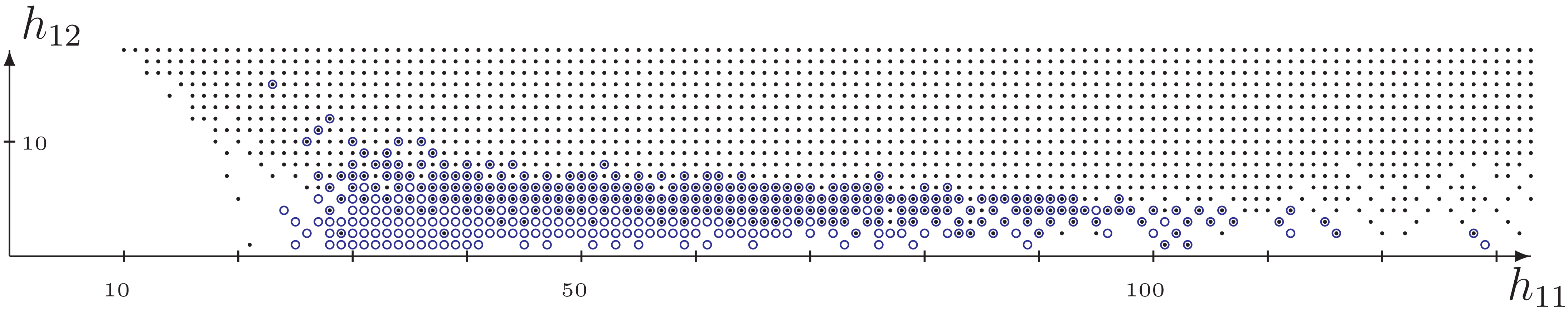
$$h_{11} = l(\Delta^*) - 1 - d - \sum_{cd(\theta^*)=1} l^*(\theta^*) + \sum_{cd(\theta^*)=2} l^*(\theta^*) l^*(\theta)$$

for $r > 1$ a combinatorial characterization of intersecting divisors is missing !

New CY mirror pairs from conifold transitions

w/ V. Batyrev arXiv:0802.3376 [math.AG]

- toric hypersurfaces $\longrightarrow$ curve of conifolds in ambient space $\longrightarrow$ 2-faces
- don't triangulate $\Rightarrow$ inherit (generically) isolated conifold singularities
- 473 800 776 $\supseteq$ 198 849 polytopes: $\forall$ 2-faces are unimod Δ or minimal $\Box$
- Smoothing condition: 30 241 mirror pairs (?) ... conjectured/suggest: flat deformation [Namikawa] $\overset{MS}{\longleftrightarrow}$ symplectic surgery [Smith,Thomas,Yau]



$h_{11} = 1$: 8871 CYs with h12 = 21,23-51,53,55,59,61,65,73,76,79,89,101,103,129
210 smooth: h12 = 25,28-41,45,47,51,53,55,59,61,65,73,76,79,89,101,103,129

$h_{11} = 2$: 43080 CYs with h12 = 22,24-80,82-90,96,100,102,103,111,112,116,128
3470 smooth: h12 = 26,28-60,62-68,70,72,74,76,77,78,80,82-84,86,88,90,96,100,102,112,116,128

$h_{11} = 3$: ...

h_{11}	$\#(\Delta)_C$	$\#(\Delta)_H$	$\#(Euler)_C$	$\#(Euler)_H$	#(diffeo. types)
1	210	5	30	5	69
2	3470	36	60	18	
3	11389	244	68	42	
4	10264	1197	72	87	
5	3808	4990	66	113	
6	815	17101	47	128	
7	140	50376	26	149	
8	35	128165	10	158	
9	3	...			

Picard number $h_{11} = 1$

- Thm. (C.T.C. Wall): diffeomorphism type $\leftrightarrow$ tripple intersections and linear form $c_2 \cdot H_i$ (for torsion-free cohomology)
- 210 polytopes $\rightarrow$ 69 diffeomorphism types with 30 Euler numbers
- We computed 30 PF operators (of 109)
  - up to 13 different polytopes / CY
  - up to 5 different principal periods / CY !!!

  Conjecture [hep-th/0410018]: equal instanton numbers, PF operators related by rational transformations (so far verified in all computable cases)

Picard Fuchs operators: $\theta = t\,\frac{d}{dt}$

$$
\begin{aligned}
&\theta^4 + \frac{2}{29}\, t\, \theta(24\theta^3 - 198\theta^2 - 128\theta - 29) - \frac{4}{841}\, t^2\, (44284\theta^4 + 172954\theta^3 + 248589\theta^2 + 172057\theta + 47096) \\
&\quad - \frac{4}{841}\, t^3\, (525708\theta^4 + 2414772\theta^3 + 4447643\theta^2 + 3839049\theta + 1275594) \\
&\quad - \frac{8}{841}\, t^4\, (1415624\theta^4 + 7911004\theta^3 + 17395449\theta^2 + 17396359\theta + 6496262) \\
&\quad - \frac{16}{841}\, t^5\, (\theta + 1)(2152040\theta^3 + 12186636\theta^2 + 24179373\theta + 16560506) \\
&\quad - \frac{32}{841}\, t^6\, (\theta + 1)(\theta + 2)(1912256\theta^2 + 9108540\theta + 11349571) \\
&\quad - \frac{10496}{841}\, t^7\, (\theta + 1)(\theta + 2)(\theta + 3)(56710 + 16301) - \frac{24529152}{841}\, t^8\, (\theta + 1)(\theta + 2)(\theta + 3)(\theta + 4)
\end{aligned}
$$

- The 210 polytopes for 1-parameter CYs have **up to 28 vertices!**
- The PF operators are mostly (except for 3) in the list of CY-equations by [G. Almkvist, C. van Enckevort, D. van Straten, W. Zudilin]

ToDo

- Compute topologies and PF operators for small h_{11}
- List Reflexive Gorenstein Cones ($\Rightarrow$ CICYs): $\begin{cases} \text{for small codim.!} \\ \longrightarrow \text{stabilization ?} \end{cases}$
- Improve program interfaces: PALP $\in$ SAGE (Softw.Alg.Geom.Experiment.)

ToFindOut

- Combinatorics/geometry dictionary for CICYs and Conifold-CYs
  ... including integral cohomology
- Study conifold transitions in CICYs and other singularities, ...

ToApply

- Use for phenomenology
- Join the effort !