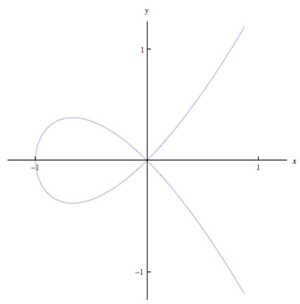
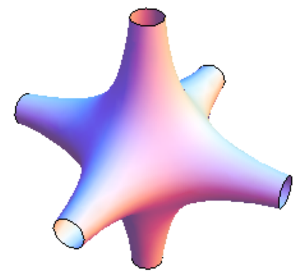
Math 480, Computational Algebraic Geometry

Description

Certain geometric objects may be represented as solutions to polynomial equations. For example, the first picture below is the set of points (x,y) in the plane that satisfy the equation $y^2 - x^2(x-1) = 0$, while the second picture is the set of points (x,y,z) in Euclidean three-space that solve the equation $x^2y^2 + y^2z^2 + z^2x^2 - 1/8 = 0$.



$y^2 - x^2(x-1) = 0.$



$x^2y^2 + y^2z^2 + z^2x^2 - 1/8 = 0$

Algebraic geometry builds a dictionary between abstract algebra and certain geometric objects. This course will provide an introduction to elementary algebraic geometry and some computational techniques that utilize it. We will learn to translate questions about solutions of polynomial equations to questions about geometry and vise-versa. In particular, we will discuss systems of polynomial equations (ideals), their solutions (varieties), and explicit computational techniques (algorithms).

Programming experience is not necessary, but you will be required to use a computer algebra system to do some of the homework assignments. You must also know how to write proofs on the level of Math 202/203. I will try to emphasize computational applications, but theorems and proofs will not be neglected.

Outline

We will attempt to cover the following topics over the course of the semester.

1. Polynomial rings, affine varieties, and ideals
2. Gröbner bases and Buchberger's algorithm
3. The elimination and extension theorems
4. The relationship between algebra and geometry (the Nullstellensatz)
5. Functions on a variety
6. Projective varieties
7. Advanced topics

Possible topics to cover near the end of the course

- Invariant theory of finite groups
- Computer vision
- Automatic theorem proving

Goals

By the end of the course, you should be able to:

- Determine if a polynomial is in a given ideal
- Determine if two systems of equations describe the same geometric object
- Understand the relationship between these two statements (Hilbert's Nullstellensatz)
- Find common solutions for a collection of polynomials over an arbitrary field
- Understand and apply Buchberger's algorithm for computing Gröbner bases
- Prove the Elimination and Extension theorems
- Parametrize systems of polynomial equations
- Apply Gröbner bases to real-world problems