

Math 5xx: Inequalities, Geometry, and the Calculus of Variations

Short description: One of the most famous analytic inequalities of all time is the arithmetic/geometric mean inequality: for any set of numbers (a_1, \dots, a_n)

$$(a_1 + \dots + a_n)/n \geq (a_1 \times \dots \times a_n)^{1/n}$$

with equality only when the numbers are all equal. A similarly famous geometric inequality is the isoperimetric inequality: for any closed curve in the plane, the ratio

Area/Length

is maximized for the round circle. Inequalities such as these are powerful tools in the study of geometry and analysis. This course presents a systematic survey of the basic theory of inequalities, leading up to applications in problems of geometric optimization. We will end with a survey of various forms of the isoperimetric inequality, and some unexpected results of these theorems, including a proof of the Faber-Krahn theorem: Among all domains with unit area in the plane, the round circle has the lowest first eigenvalue of the Laplacian.

Textbook: Hardy, Littlewood and Polya. *Inequalities*. (\$44, paperback)

Recommended Reading: Gelfand and Fomin. *Calculus of Variations* (\$8, paperback)

Short Syllabus:

Elementary Mean Value Theorems

Cauchy's Inequality

Holder's Inequality

Minkowski's Inequality

Application: The polynomial $x^k(x-y)(x-z) + y^k(y-x)(y-z) + z^k(z-x)(z-y) > 0$ when $x, y, z, k > 0$.

Weighted Mean Value Theorems

Theory of convex functions

Jensen's Inequality

Application: Polygons inscribed in a circle

Calculus and Inequalities

Infinite Series Inequalities

Integral Inequalities

Holder's Inequality for integrals

Minkowski's Inequality for integrals

Application: The kinetic energy of a fluid

The Calculus of Variations

General Ideas

Wirtinger's Inequality

Application: The average chord length of a closed curve in the plane

Isoperimetric Inequalities for Curves

The coarea formula

Application: The shape of the drum with the lowest note (The Faber-Krahn Theorem)