Math 241 Maple Project

Modeling the Temperature Distribution of Blood Using the Cauchy-Euler and Bessel Equations

Introduction

The goal of this project is to estimate the temperature distribution in blood as it flows through the veins. The veins are assumed to be infinitely long cylinders. Thus, the temperature distribution in the blood $v(r, \theta, z)$, will satisfy the equation of conduction in cylindrical coordinates:

$$\frac{\partial v}{\partial t} = c^2 \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial v}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 v}{\partial \theta^2} + \frac{\partial^2 v}{\partial z^2} \right] \tag{1}$$

where r and θ are the polar coordinates of the temperature of the blood in a cross section of the vein, taken at some value of the height variable, z. It is difficult to solve equations in this generality. In this project, we will model this equation under various conditions as outlined below. Typically, the equation will be of a different known type (Fourier-Bessel, Cauchy-Euler, etc.).

A general assumption when modeling the heat distribution in blood is to assume the absence of clotting or *thrombosis*. Internal thrombosis in the heart or blood vessels is dangerous to life, however if blood leaves the blood vessel and comes into contact with a foreign surface, the necessary coagulation process is immediately initiated which begins to repair the wound allowing for normal functionality once the wound has healed.

Blood is a highly complex fluid that nutures life and contains many enzymes and hormones that are essential for normal functionality. Blood transports oxygen and carbon dioxide between the lungs and cells of the tissues. Modeling the temperature distribution of blood is a problem of great interest in bioengineering. As with many mathematical models of biological processes, modeling the heat distribution of blood is a difficult task due to the constant motion and exchange of blood and due to external changes in temperature. These complication require engineers to make several assumptions regarding the flow of blood, so that the model can produce reasonable output with a minimal amount of computational time.

If the blood in the vein is exposed to heat, and if the boundary conditions are assumed to be independent of the coordinates θ and z, then the temperature distribution in the blood is a function of r and t only, so that $v = v(r, t)$. The governing partial differential equation of the heat distribution reduces to:

$$\frac{\partial v}{\partial t} = c^2 \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial v}{\partial r} \right) \tag{2}$$

In this case, the flow of heat in the blood takes place in planes perpendicular to the z-axis, indicating that the lines of heat flow are radial. With the given simplifications, the solution is of the standard *Fourier-Bessel* type. Further simplifications are made if the flow of heat is assumed to be steady state.

Questions

Part A: Steady Temperature. Radial Heat Flow.

1. Find the steady state temperature of blood in a vein assuming that the flow of heat is radial, so that the temperature of the blood depends only on r. Assume that the inner radius of the vein is a and the outer radius is b. Show that the governing equation reduces to a Cauchy-Euler equation, and write the answer in terms of arbitrary constants c_1 and c_2.

2. Assuming that at $r = a$, the temperature is held fixed at v_1 and at $r = b$, the temperature is held fixed at v_2, use Maple to determine the heat distribution of the blood in terms of the variable r and constants a, b, v_1, v_2.

3. The flow of heat per unit length in the blood between $r = a$ and $r = b$ is proportional to the difference of temperature: $v_1 - v_2$. Use Maple to determine the rate of heat flow in the blood and show that it is proportional to $v_1 - v_2$. The rate of heat flow is defined as:

$$-2\pi r K \frac{dv}{dr}$$

   where the constant K is the thermal conductivity of blood.

4. Now, assume that at $r = a$, the temperature is held fixed at v_1, but at $r = b$, there is radiation of heat from a nearby artery into a medium at v_2. The boundary condition at $r = b$ is given by:

$$\frac{dv}{dr} + h(v - v_2) = 0$$

   where h is a constant that is proportional to the coefficient of surface heat transfer.

   Use Maple to determine the heat distribution of the blood in terms of the variable r and constants a, b, v_1, v_2, h.

5. Find the outward rate of flow of heat (as defined in problem #3) in the blood per unit length of the blood vessel. Assuming that K is unity, v_1 is 100 heat units and v_2 is 95 heat units, set $ah > 1$ and plot the flow of heat as r increases from a to b. Does the flow of heat increase or decrease over this region? What is the physical interpretation in terms of heat distribution in the blood?

   Set $ah < 1$ and plot the flow of heat as r increases from a to b. Does this graph steadily increase or decrease? Are there any maximum or minimum points? If so, find them. What is the significance of a maximum or minimum point in terms of the physical interpretation?

Part B: Unsteady Temperature. Radial Heat Flow.

A highly accurate model of blood temperature would take into account the proximity of the major arteries and veins to each other. This close proximity permits considerable heat exchange between artery and vein. Cold blood that flows in veins from a cooled extremity toward the heart is heated during transport from the warmer blood in the adjacent arteries, resulting in warming of the venous blood and cooling of the arterial blood. Heat exchange occurs in the opposite direction when extremities of the body are exposed to heat. This enables the body to enhance heat conservation during exposure to extreme cold temperatures and minimize heat gain when exposure to warm environments occurs.

Suppose that blood is flowing towards the heart through a long portion of a vein with unit radius, such that initially, the temperature of the blood at all points along the vein is 65 heat units. Assume that the blood is being transported from only one source. Further, at $t = 0$ assume that the blood is re-directed through a different vein, and that the surface temperature of this new vein is maintained at 50 heat units for $t > 0$. Assuming that the temperature distribution in the blood does not depend on the z variable, find the temperature in the new vein as a function of r and t. Proceed to solve this problem by following the steps outlined below:

1. State the partial differential equation, boundary conditions and initial condition that represent this problem. State the assumptions that will be necessary in order to obtain a solution.

2. By using the method of separation of variables, reduce the modified conduction equation (2) to two differential equations. Solve each differential equation to obtain the basic solutions. One of the answers contains Bessel's function of the second kind, $Y_0(r)$. Use Maple to plot $Y_0(r)$ and verify that it is not bounded on the interval $r \in [0, 1]$. Reduce your answer so that the solutions remain bounded on $r \in [0, 1]$. Show that these solutions tend to zero as $t \to \infty$.

3. To the basic solutions in problem #2, add the final steady state temperature and verify that $v(r, t) \to 50$ as $t \to \infty$.

4. In order for the temperature of the blood to be 50 heat units at $r = 1$, it is required that the basic solutions are equal to zero when $r = 1$. It is important to be able to determine the values which force the basic solutions to vanish when $r = 1$. Proceed to determine several of these values by using Maple to plot $J_0(r)$ on the interval $r \in [0, 16]$. Determine numerically the values of the first twenty-five positive λ_n values such that $J_0(\lambda_n) = 0$ (These are the eigenvalues of the Sturm-Liouville problem).

5. Write the solution to the boundary value problem in summation notation and in terms of arbitrary constants c_n. Use Maple to evaluate the first five nonzero c_n terms. Construct a three dimensional patch and contour plot of the temperature distribution in the blood as a function of r and t. Label all axes and indicate the boundary conditions on the graph.

6. Construct a new graph of the temperature of the blood vs. r and t, but this time use the first twenty-five terms in the series solution. What are the major differences between the graph that uses the first five terms and the one that uses the first twenty-five terms? What (if any) differences would you expect to obtain if you were to plot the temperature distribution using the first fifty nonzero terms of the series solution?

7. How long does it take for the temperature of the blood to decrease to 50 heat units?

8. Are the results limited by any of the assumptions that you've made? Explain. How would the results change if each of the assumptions was not made?