dsolve

The basic Maple command for solving differential equations is "`dsolve`". The basic syntax of dsolve is the usual

```
dsolve("what", "how");
```

syntax of most basic Maple commands. "What" refers to the differential equation (or system of differential equations) together with any initial conditions there might be -- if there is more than just one equation with no conditions, this must be enclosed in braces {}. The "How" part, like most solve routines in Maple, indicates the name of the variable or function to be solved for.

It is useful to give names to all of the equations and initial conditions you are going to use in `dsolve` -- it makes the statements easier to read and can often save some typing. For example:

```
> eq:=diff(y(x),x)=x*y(x);
```

$$eq := \frac{\partial}{\partial x} y(x) = x \, y(x)$$

```
> init:=y(2)=1;
```

$$init := y(2) = 1$$

Now, eq is the name of the differential equation we will solve, and `init` is the name of the initial condition. It is important that we use `y(x)` rather than just `y` -- this indicates to Maple that we are thinking of `y` as the dependent variable and `x` as the independent one. To solve the equation WITHOUT the initial condition (i.e., to find the general solution), we

```
> dsolve(eq,y(x));
```

$$y(x) = e^{(1/2 \, x^2)} \_C1$$

Notice the _C1 -- that is Maple's way of producing an "arbitrary constant". To solve the initial-value problem, we must group the equation and initial condition together in braces:

> **dsolve({eq,init},y(x));**

$$y(x) = \frac{e^{(1/2\,x^2)}}{e^2}$$

This last output is an equation -- if you wish to assign the output to a name, so that you can use it for further work (or to plot it, etc..) it is possible to use the "rhs" (right-hand side) command (and the double quote, which refers to the last previous output):

> **ans:=rhs(");**

$$ans := \frac{e^{(1/2\,x^2)}}{e^2}$$

Sometime Maple will give the answer in an "implicit" form -- an equation that relates the dependent and independent variables instead of just an expression for the independent variable:

> **dsolve(diff(y(x),x)=y(x)^2, y(x));**

$$\frac{1}{y(x)} = -x + \_C1$$

But it is clear that one can solve easily for y(x) -- to force Maple to do this, use the " explicit " option in dsolve:

> **dsolve(diff(y(x),x)=y(x)^2, y(x), explicit);**

$$y(x) = -\frac{1}{x - \_C1}$$

That's better.

More advanced uses of dsolve:

There are five additional ways to use `dsolve` for situations other than single first-order differential equations and initial-value problems.

1. **Equations of higher order** : Second (and higher) order equations can be solved with `dsolve`. To do this, recall that higher derivatives of a function are taken as follows in Maple (this is the third derivative of f(x)):

```
> diff(f(x),x$3);
```

$$\frac{\partial^3}{\partial x^3} f(x)$$

So, we can set up a second-order equation as follows:

```
> eqn2:=diff(y(x),x$2)+3*diff(y(x),x)+2*y(x)=exp(x);
```

$$eqn2 := \left(\frac{\partial^2}{\partial x^2} y(x)\right) + 3\left(\frac{\partial}{\partial x} y(x)\right) + 2\, y(x) = e^x$$

The general solution of this equation is:

```
> dsolve(eqn2,y(x));
```

$$y(x) = \frac{1}{6} e^x + \_C1\, e^{(-2x)} + \_C2\, e^{(-x)}$$

Notice that there are two constants, _C1 and _C2 (as to be expected) in the solution. The appropriate initial-value problem requires two initial conditions -- one on the value of y at some value of x, and the other on the value of the derivative at the point. For this problem, we will specify values of y at x=1:

```
> inits:=y(1)=2, D(y)(1)=4;
```

$$inits := y(1) = 2, D(y)(1) = 4$$

```
> dsolve({eqn2,inits},y(x));
```

$$y(x) = \frac{1}{6} \frac{(e^x)^2 + 6\left(\frac{1}{3}(e)^3 - 6(e)^2\right)e^{(-2x)}e^x - 3(e)^2 + 48\,e}{e^x}$$

Sometimes these things get somewhat complicated!

2. **Systems of differential equations** : In many applications, systems of differential equations arise. The syntax for these is the same as for initial-value problems (even when no initial values are specified) --- braces around the problem are required, AND braces around the list of functions to be solved for. We do one example of an initial value problem for a system of two first-order equations:

```
> eqns:=diff(y(x),x)+diff(z(x),x)=x,
  diff(y(x),x)-2*diff(z(x),x)=x^2;
```

$$eqns := \left(\frac{\partial}{\partial x} y(x)\right) + \left(\frac{\partial}{\partial x} z(x)\right) = x, \left(\frac{\partial}{\partial x} y(x)\right) - 2\left(\frac{\partial}{\partial x} z(x)\right) = x^2$$

```
> inits:= y(0)=1, z(0)=2;
```

$$inits := y(0) = 1, z(0) = 2$$

```
> dsolve({eqns,inits},{y(x),z(x)});
```

$$\{ y(x) = 1 + \frac{1}{3}x^2 + \frac{1}{9}x^3, z(x) = 2 + \frac{1}{6}x^2 - \frac{1}{9}x^3 \}$$

3. **Numerical solutions**: Very often, it is impossible to obtain the solution to a differential equation in closed form. In this case, one must resort to a numerical approximation method. Maple knows a lot of these -- to invoke them, use the " `numeric` " option of `dsolve` , as follows:

```
> eqn:=diff(y(x),x)+exp(y(x))*x^3=2*sin(x); init:=y(0)=2;
```

$$eqn := \left(\frac{\partial}{\partial x} y(x)\right) + e^{y(x)} x^3 = 2 \sin(x)$$

$$init := y(0) = 2$$

```
> F:=dsolve({eqn,init},y(x),numeric);
```

$$F := \textbf{proc}(\textit{rkf45\_x}) \ ... \ \textbf{end}$$

Your output for the preceeding statement might be slightly different. The output means that Maple has defined a function called `F`, that will output the value of `y(x)` corresponding to a given (numerical) value of `x`. For example:

```
> F(2);
```

$$[x = 2, \ y(x) = -.7815970926289340]$$

It is often useful to plot the numerically-obtained solution of a differential equation. To do this, you need to apply the Maple command `odeplot` to the result (`F` in this case) of the `dsolve(....,numeric)` statement. The `odeplot` command is in the `plots` library and so must be loaded using the statement:
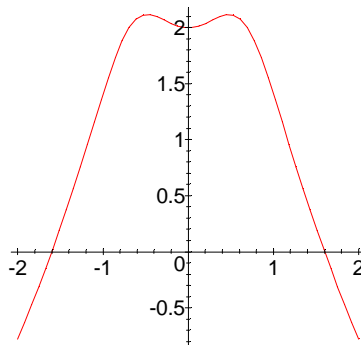
```
> with(plots,odeplot);
```

$$[\textit{odeplot}]$$

To plot the solution, use the following syntax:

```
> odeplot(F,[x,y(x)],-2..2);
```


In this statement, the "`F`" is the function that resulted from `dsolve(...numeric)`. The second argument indicates to Maple which variables should be on which axis (sometimes it is useful to plot x versus the derivative of y(x) or y(x) versus the derivative of y(x)). The final axis gives the domain of the independent variable (x in this case) over which the plot should be made.

The most important things to notice are:

   (a) You must assign the result of `dsolve(......,numeric)` to a NAME.

   (b) You must use the `odeplot` function from the `plots` library to plot the result of `dsolve(......,numeric)`. The syntax is as above, and you can see some more examples in Maple help.

4. You can also use dsolve to get **power series solutions of differential equations**. It is usually best to specify an initial-value problem to do this. The order to which the series is computed is determined by the system variable `Order` (just as for the `taylor` command). For example, to solve the initial value problem y'+x*y=0, y(0)=1 by series, you enter:

```
> dsolve({diff(y(x),x)+x*y(x)=0, y(0)=1},y(x), series);
```

$$y(x) = 1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 + O(x^6)$$

If you want more terms:

```
> Order:=14: dsolve({diff(y(x),x)+x*y(x)=0, y(0)=1},y(x), series);
```

$$y(x) = 1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 - \frac{1}{48}x^6 + \frac{1}{384}x^8 - \frac{1}{3840}x^{10} + \frac{1}{46080}x^{12} + O(x^{14})$$

The result of this operation is a Maple "series" (with the "big-O" term) -- see the discussion of this in the `taylor` section of this manual.

5. (for Math 241): `dsolve` can also be used to solve differential equations by **Laplace transform methods**. The setup is similar to getting power series solutions as above, except instead of using the option "`series`", we use "`method=laplace`". For example:

```
> dsolve({diff(y(x),x)+y(x)=Dirac(x-1),y(0)=0},
  y(x),method=laplace);
```

$$y(x) = \text{Heaviside}(x - 1)\, e^{(-x + 1)}$$

The Dirac and Heaviside functions are the Dirac delta and unit step functions, respectively (see Chapter 4 of the Math 241 textbook). For more information about Laplace transforms in Maple, see the section on the `laplace` command.

Errors:

The most common error one makes when using dsolve is to use the dependent variable (the "y" above) without specifying the dependence on the independent variable (in other words, using the `y` without writing `y(x)`). Unexpected results often happen when you do this.

The other kinds of errors are like the ones that go wrong whenever you use a solving or plotting routine -- e.g., the variables being solved for have already been assigned values (that may have been forgotten), etc..