Math 312, Midterm 2

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You have 50 minutes to complete this midterm.

If n is a positive integer, let

$$\mathcal{P}_n := \{f(x) \in \mathbb{R}[x] \mid \deg f \leq 50\}$$

be the vector space of polynomials of dimension n. In particular, any particular element of $\mathcal{P}_n$ is a polynomial in the variable x. Let

$$\mathcal{P}_n^* := \{\varphi : \mathcal{P}_n \to \mathbb{R} \mid \varphi \text{ linear}\}.$$

denote the dual space of $\mathcal{P}_n$. We have an inner product on $\mathcal{P}_n$ given by

$$\langle f, g \rangle := \int_0^1 f(x)g(x)\, dx.$$

Consider the linear function

$$D : \mathcal{P}_{50} \to \mathcal{P}_{50}$$

given by

$$D(f) = f'(x),$$

the derivative of the polynomial f.

1. **(5 points)** What is the dimension of $\mathcal{P}_{50}$?

2. **(5 points)** What is the dimension of $\mathcal{P}_{50}^*$?

3. **(10 points)** What is $\dim \ker D$?

4. **(10 points)** What is $\dim \operatorname{im} D$?

5. Consider the following subsets of $\mathcal{P}_{50}$. Which are vector spaces, under the induced operations of addition and scalar multiplication? Which are not? Give a brief answer for each.

   (a) **(10 points)** $A = \{f(x) \in \mathcal{P}_{50} \mid \langle f(x), f(x) \rangle = 1\}$.

   (b) **(10 points)** $B = \{f(x) \in \mathcal{P}_{50} \mid \langle f(x), f(x) \rangle \geq 1\}$.

(c) **(10 points)** $C = \{f(x) \in \mathcal{P}_{50} \mid \langle f(x), f(x) \rangle \geq 0\}$.

(d) **(10 points)** $D = \{f(x) \in \mathcal{P}_{50} \mid \langle f(x), x^3 + x^2 + 1 \rangle = 0\}$.

(e) **(10 points)** $E = \{f(x) \in \mathcal{P}_{50} \mid \langle f(x), x^3 + x^2 + 1 \rangle = 1\}$.

(f) **(10 points)** $F = \{f(x) \in \mathcal{P}_{50} \mid f(0) = f(1) = 1\}$.

(g) **(10 points)** $G = \{f(x) \in \mathcal{P}_{50} \mid f(0) = 0, f(1) = 1\}$.

6. Let

$$H = \{f(x) \in \mathcal{P}_{50} \mid f(0) = f(1) = 0\}.$$

H is a vector space.

(a) **(10 points)** What is $\dim H$?

(b) **(15 points)** Let $g \in \mathcal{P}_{50}$. Find the unique $D^*g \in \mathcal{P}_{50}$ such that for all $f \in H$,

$$\langle Df, g \rangle = \langle f, D^*g \rangle.$$

(hint: integration by parts. Don't forget to use all hypotheses!)