# Problem Set 3

Due: To Shanshan's office on Friday, Feb. 1 at 1 pm.

1. Let $A$ and $B$ both be $n \times n$ matrices. What's wrong with the formula $(A+B)^2 = A^2 + 2AB + B^2$? Prove that if this formula is true for $A$ and $B$, then $A$ and $B$ commute.

2. Which of the following subsets of $\mathbb{R}^2$ are actually subspaces? Explain.
   a) $\{(x,y) \mid xy = 0\}$
   b) $\{(x,y) \mid x \text{ and } y \text{ are both integers}\}$
   c) $\{(x,y) \mid x+y = 0\}$
   d) $\{(x,y) \mid x+y = 2\}$
   e) $\{(x,y) \mid x+y \geq 0\}$

3. Let $V$ and $W$ be linear spaces and $T : V \to W$ a linear map.
   a) Assume the kernel of $T$ is trivial, that is, the only solution of the homogeneous equation $T\vec{x} = 0$ is $\vec{x} = 0$. Prove that if $T(\vec{x}) = T(\vec{y})$, then $\vec{x} = \vec{y}$.
   b) Conversely, if $T$ has the property that "if $T(\vec{x}) = T(\vec{y})$, then $\vec{x} = \vec{y}$," show that the kernel of $T$ is trivial.

4. Say $\vec{v}_1, \ldots, \vec{v}_n$ are linearly independent vectors in $\mathbb{R}^n$ and $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear map.
   a) Show by an example, say for $n = 2$, that $T\vec{v}_1, \ldots, T\vec{v}_n$ need not be linearly independent.
   b) However, show that if the kernel of $T$ is trivial, then these vectors $T\vec{v}_1, \ldots, T\vec{v}_n$ are linearly independent.

5. Let $A : \mathbb{R}^3 \to \mathbb{R}^5$ and $B : \mathbb{R}^5 \to \mathbb{R}^2$.
   a) What are the maximum and minimum values for the dimension of the kernels of $A$, $B$, and $BA$?
   b) What are the maximum and minimum values for the dimension of the images of $A$, $B$, and $BA$?

6. [Bretscher, Sec. 2.4 #52]. Let $A := \begin{pmatrix} 0 & 1 & 2 \\ 0 & 2 & 4 \\ 0 & 3 & 6 \\ 1 & 4 & 8 \end{pmatrix}$. Find a vector $\vec{b}$ in $\mathbb{R}^4$ such that the system $A\vec{x} = \vec{b}$ is inconsistent, that is, it has no solution.

7. Find a real $2 \times 2$ matrix $A$ (with $A^2 \neq I$ and $A^3 \neq I$) so that $A^6 = I$. For your example, is $A^4$ invertible?

8. Let $A$, $B$, and $C$ be $n \times n$ matrices with $A$ and $C$ invertible. Solve the equation $ABC = I - A$ for $B$.

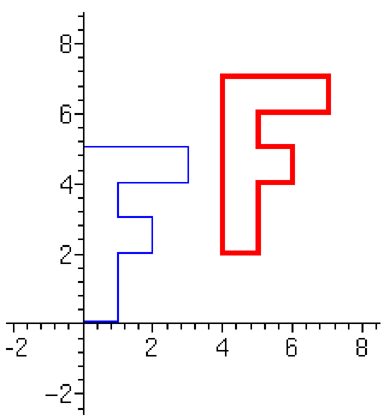
9. If a square matrix $M$ has the property that $M^4 - M^2 + 2M - I = 0$, show that $M$ is invertible. [SUGGESTION: . Find a matrix $N$ so that $MN = I$. This is very short.]

10. Linear maps $F(X) = AX$, where $A$ is a matrix, have the property that $F(0) = A0 = 0$, so they necessarily leave the origin fixed. It is simple to extend this to include a translation,

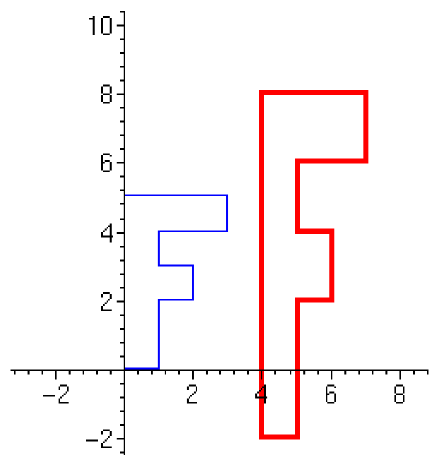
$$F(X) = V + AX,$$

where $V$ is a vector. Note that $F(0) = V$.

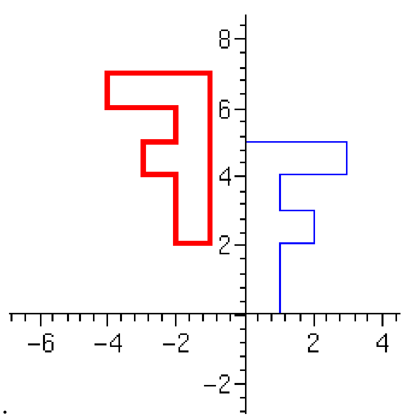
Find the vector $V$ and the matrix $A$ that describe each of the following mappings [here the light blue $F$ is mapped to the dark red $F$].



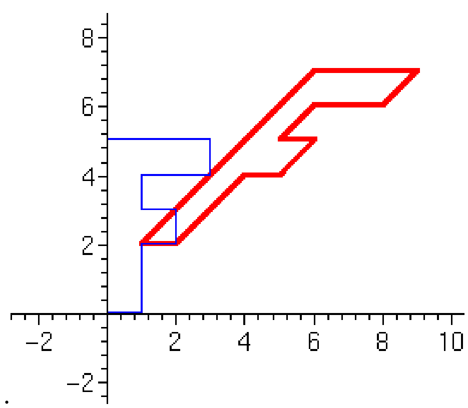
a).



b).



c).



d).

11. Let $\vec{e}_1 = (1, 0, 0, \dots, 0) \in \mathbb{R}^n$ and let $\vec{v}$ and $\vec{w}$ be any non-zero vectors in $\mathbb{R}^n$.
   a) Show there is an invertible matrix $B$ with $B\vec{e}_1 = \vec{v}$.
   b) Show there is an invertible matrix $M$ with $M\vec{w} = \vec{v}$.

12. [Like Bretscher, Sec. 2.4 #40]. Let $A$ be a matrix, not necessarily square.
   a) If $A$ has two equal rows, show that it is not onto (and hence not invertible).
   b) If $A$ has two equal columns, show that it is not one-to-one (and hence not invertible).

[Last revised: January 26, 2013]