Math 312, Spring 2013 Aaron M. Silberstein

Problem Set 4

DUE: To Shanshan's mailbox in the math office, 1 p.m., Feb. 8.

Reminder: Exam 1 is on Wednesday, Feb. 13, 11:00-11:50. No books or calculators but you may always use one $3'' \times 5''$ card with handwritten notes on both sides.

1. Let A be a square matrix. If A^2 is invertible, show that A is invertible.

2. Find a linear map $L : \mathbb{R}^3 \to \mathbb{R}^3$ whose kernel is *exactly* the plane

$$\{(x_1,\, x_2,\, x_3) \subset \mathbb{R}^3 \mid x_1 + 2x_2 - x_3 = 0\}.$$

3. In class we considered the interpolation problem of finding a polynomial of degree n passing through $n+1$ specified distinct points in the plane. To be definite, take $n = 3$, and say we want to find a polynomial ϕ which passes through (a_1, b_1), (a_2, b_2), (a_3, b_3), and (a_4, b_4), with $a_i \neq a_{i'}$ if $i \neq i'$. The point of this problem is to see vividly why choosing a basis adapted to the problem may involve much less work. Recall that $\mathcal{P}_n$ is the vector space of polynomials of degree $\leqslant n$. Let

$$\mathrm{ev}_{a_i} : \mathcal{P}_3 \to \mathbb{R}$$

be the element of $\mathcal{P}_3^*$ given by

$$\mathrm{ev}_{a_i}(f(x)) = f(a_i).$$

a) Prove that $B = \{\mathrm{ev}_{a_1}, \mathrm{ev}_{a_2}, \mathrm{ev}_{a_3}, \mathrm{ev}_{a_4}\}$ is a basis of $\mathcal{P}_3^*$.

b) Compute the dual basis $B^* = \{\beta_1, \beta_2, \beta_3, \beta_4\} \subset \mathcal{P}_3^{**} = \mathcal{P}_3$ of B.

c) What is a polynomial ϕ whose graph passes through the points (a_i, b_i), as a linear combination of the β_i?

d) Why is there exactly one such ϕ?

This basis $\{\beta_i\}$ is called the **Lagrange basis** for this interpolation problem.

4. [BRETSCHER, SEC. 2.4 #35] An $n \times n$ matrix A is called *upper triangular* if all the elements below the *main diagonal*, a_{11} a_{22}, $\ldots$ a_{nn} are zero, that is, if $i > j$ then $a_{ij} = 0$.

a) Let $V_i \subseteq \mathbb{R}^n$ be the span of $\{e_1, \ldots, e_i\}$. Prove (using induction) that a matrix A is upper triangular if and only if $A(V_i) \subseteq V_i$. That is, Ae_1 is a multiple of e_1; Ae_2 is a linear combination of e_1 and e_2; *et cetera*.

b) Let A be the upper triangular matrix

$$A = \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix}.$$

For which values of a, b, c, d, e, f is A invertible? HINT: Write out the equations $AX = Y$ explicitly.

c) If A is invertible, is its inverse also upper triangular?

d) Show that the product of two $n \times n$ upper triangular matrices is also upper triangular.

e) Show that an upper triangular $n \times n$ matrix is invertible if none of the elements on the main diagonal are zero.

f) Conversely, if an upper triangular $n \times n$ matrix is invertible show that none of the elements on the main diagonal can be zero.

5. [SEE BRETSCHER, SEC. 3.2 #6] Let U and V both be two-dimensional subspaces of $\mathbb{R}^5$, and let $W = U \cap V$. Find all possible values for the dimension of W.

6. [SEE BRETSCHER, SEC. 3.2 #50] Let U and V both be two-dimensional subspaces of $\mathbb{R}^5$, and define the set $W := U + V$ as the set of all vectors $w = u + v$ where $u \in U$ and $v \in V$ can be any vectors.

   a) Show that W is a linear space.

   b) Find all possible values for the dimension of W.

7. Say you have k linear algebraic equations in n variables; in matrix form we write $A\vec{x} = \vec{y}$. Give a proof or counterexample for each of the following.

   a) If $n = k$ there is always *at most one* solution.

   b) If $n > k$ you can *always* solve $A\vec{x} = \vec{y}$.

   c) If $n > k$ the nullspace (= kernel) of A has dimension greater than zero.

   d) If $n < k$ then for *some* $\vec{y}$ there is *no* solution of $A\vec{x} = \vec{y}$.

   e) If $n < k$ the *only* solution of $A\vec{x} = 0$ is $\vec{x} = 0$.

8. [BRETSCHER, SEC. 3.3 #30] Find a basis for the subspace of $\mathbb{R}^4$ defined by the equation $2x_1 - x_2 + 2x_3 + 4x_4 = 0$.

9. Let V the vector space of $n \times n$ matrices A with real entries. Define a transformation $T : V \to V$ where $T(A) = \frac{1}{2}(A + A^T)$. (Here, A^T is the matrix transpose of A.)

a) Verify that T is linear. You may use familiar facts about transpose.

b) Describe the image of T , and find its dimension.

c) Describe the kernel of T , and find its dimension.

d) Verify the rank and nullity add up what you would expect. (Final note: T is called the *symmetrization operator*.)

10. Recall that $\mathcal{P}_2$ be the linear space of polynomials of degree at most 2, and let $T : \mathcal{P}_2 \to \mathcal{P}_2$ be the transformation

$$(T(p))(t) = \frac{1}{t}\int_0^t p(s)\,ds.$$

For instance, if $p(t) = 2 + 3t^2$, then $T(p) = 2 + t^2$.

a) Prove that T is a linear transformation.

b) Find the kernel of T, and find its dimension.

c) Find the range (=image) of T, and compute its dimension.

d) Verify the dimension of the kernel and the dimension of the range add up to what you would expect.

e) Using the standard basis $\{1, t, t^2\}$ for $\mathcal{P}_2$, represent the linear transformation T as a matrix A.

f) Using your matrix represention from (e), find $T(p)$ where $p(t) = t - 2$.

Bonus Problem

[Please give this directly to Professor Silberstein]

1-B Let $L\colon V \to V$ be a linear map on a linear space V.

a) Show that $\ker L \subset \ker L^2$ and, more generally, $\ker L^k \subset \ker L^{k+1}$ for all $k \geqslant 1$.

b) If $\ker L^j = \ker L^{j+1}$ for some integer j, show that $\ker L^k = \ker L^{k+1}$ for all $k \geqslant j$.

c) Let A be an $n \times n$ matrix. If $A^j = 0$ for some integer j (perhaps $j > n$), show that $A^n = 0$.

[Last revised: February 2, 2013]