Math 312, Spring 2013 Aaron M. Silberstein

Problem Set 5

DUE: To Shanshan's mailbox in the math department, 1 pm Friday, Feb. 22.

In addition to the problems below, you should also know how to solve the following problems from the text. Most are simple exercises. These are *not* to be handed in.

Sec. 5.1, #28, 29, 31
Sec. 5.2 #33

REMARK: We will not cover the material on QR factorization. It is an important numerical technique – but our time is short.

When we deal with the vector spaces $\mathbb{R}^n$, we will always use the unique symmetric inner product in which the standard basis $e_1, \ldots, e_n$ is orthonormal.

1. [BRETSCHER, SEC. 5.1 #16] Consider the following vectors in $\mathbb{R}^4$

$$\vec{u}_1 = \begin{pmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{pmatrix}, \qquad \vec{u}_2 = \begin{pmatrix} 1/2 \\ 1/2 \\ -1/2 \\ -1/2 \end{pmatrix}, \qquad \vec{u}_3 = \begin{pmatrix} 1/2 \\ -1/2 \\ 1/2 \\ -1/2 \end{pmatrix}.$$

   Can you find a vector u_4 in $\mathbb{R}^4$ such that the vectors $\vec{u}_1$, $\vec{u}_2$, $\vec{u}_3$, $\vec{u}_4$ are orthonormal? If so, how many such vectors are there?

2. [BRETSCHER, SEC. 5.1 #17] Find a basis for $W^\perp$, where

$$W = \text{span}\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 5 \\ 6 \\ 7 \\ 8 \end{pmatrix} \right\}.$$

3. [BRETSCHER, SEC. 5.1 #21] Find scalars a, b, c, d, e, f, and g so that the following vectors are orthonormal:

$$\begin{pmatrix} a \\ d \\ f \end{pmatrix}, \qquad \begin{pmatrix} b \\ 1 \\ g \end{pmatrix}, \qquad \begin{pmatrix} c \\ e \\ 1/2 \end{pmatrix}.$$

4. Let V be an inner product space and S a subspace. Then we write $S^\perp$ for the set of all vectors in V that are orthogonal to S. It is called the *orthogonal complement* of S, and clearly is also a subspace of V.

   a) In $\mathbb{R}^3$, let S be the points (x_1, x_2, x_3) that satisfy $x_1 - 2x_2 + x_3 = 0$. What is the dimension of $S^\perp$? [This should be a simple mental exercise.]

b) Let $A : \mathbb{R}^3 \to \mathbb{R}^5$. If the dimension of the kernel of A is 2, what is the dimension of $\text{image}(A)^\perp$?

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5. [Bretscher, Sec. 5.1 #26] Find the orthogonal projection P_S of $\vec{x} := \begin{pmatrix} 49 \\ 49 \\ 49 \end{pmatrix}$ into the subspace S of $\mathbb{R}^3$ spanned by $\vec{v}_1 := \begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix}$ and $\vec{v}_2 := \begin{pmatrix} 3 \\ -6 \\ 2 \end{pmatrix}$.

6. [Bretscher, Sec. 5.1 #37] Consider a plane V in $\mathbb{R}^3$ with orthonormal basis $\vec{u}_1$ and $\vec{u}_2$. Let $\vec{x}$ be a vector in $\mathbb{R}^3$. Find a formula for the reflection $R\vec{x}$ of $\vec{x}$ across the plane V.

7. [Bretscher, Sec. 5.2 #32] Find an orthonormal basis for the plane $\{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 + x_2 + x_3 = 0\}$.

8. [Bretscher, Sec. 5.3 #10] Consider the space $\mathcal{P}_2$ of real polynomials of degree at most 2 with the inner product

$$\langle f,\, g\rangle = \int_{-1}^{1} f(t)g(t)\,dt.$$

Find an orthonormal basis for the vector subspace of $\mathcal{P}_2$ consisting of all the degree 2 polynomials orthogonal to $f(t) = t$.

9. [Bretscher, Sec. 5.3 #16] Consider the space $\mathcal{P}_1$ with the inner product

$$\langle f,\, g\rangle = \int_{0}^{1} f(t)g(t)\,dt.$$

a) Find an orthonormal basis for this space.

b) Find the linear polynomial $g(t) = a + bt$ that best approximates the polynomial $f(t) = t^2$. Thus, one wants to pick $g(t)$ so that $\|f - g\|$ is as small as possible.

10. [See `http://www.math.upenn.edu/~kazdan/312S13/notes/Lu=-DDu.pdf`]. Consider the space $C_0^2[0,1]$ of twice continuously differentiable functions $u(x)$ with $u(0) = 0$ and $u(1) = 0$. Define the differential operator Mu by the formula $Mu = ((1+x^2)u')'$. Find the adjoint M^* (you should find that M is self-adjoint).

The remaining problems are from the Lecture notes on Vectors
`http://www.math.upenn.edu/~kazdan/312F12/notes/vectors/vectors8.pdf`.
Note the similarity in notation between adjoint matrices in these notes and dual linear transformations. This is no accident!
Recall that giving an inner product $\langle \cdot, \cdot \rangle$ on a vector space V is the same as giving a map

$$\varphi_{\langle \cdot, \cdot \rangle} : V \to V^*;$$

given the inner product, we define

$$\varphi_{\langle \cdot, \cdot \rangle}(v)(w) := \langle v, w \rangle.$$

That is, for each v, bilinearity guarantees $\varphi_{\langle \cdot, \cdot \rangle}(v)$ is a linear function of w, so is in the dual space of V. Conversely, given a linear transformation

$$\psi : V \to V^*$$

we may define an inner product

$$\langle v, w \rangle_\psi := \psi(v)(w).$$

Bilinearity follows from the assumption that ψ is a linear transformation. These are inverse constructions:

$$\varphi_{\langle v, w \rangle_\psi} = \psi.$$

Recall that **nondegeneracy** of $\langle \cdot, \cdot \rangle$ is equivalent to $\varphi_{\langle \cdot, \cdot \rangle}$ being an invertible linear map (as long as V is finite-dimensional).
As an example, given a row vector $[r_1, \ldots, r_n]$ we may define a linear transformation

$$L_{[r_1, \ldots, r_n]} : \mathbb{R}^n \to \mathbb{R}$$

from the vector space $\mathbb{R}^n$ of column vectors by:

$$L_{[r_1,\ldots,r_n]}\left(\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}\right) := [r_1, \ldots, r_n] \cdot \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \sum_{i=1}^{n} r_i c_i.$$

In other words, the row vectors may be identified with a subspace of the dual space $\mathbb{R}^{n*}$. That the row vectors, under this identification, have basis e_i^T (the transpose of the column vector associated to e_i) and $\{L_{e_1^T}, \ldots, L_{e_n^T}\}$ satisfies the properties of a dual basis to the basis $\{e_1, \ldots, e_n\}$, we see that $\mathbb{R}^{n*}$ is canonically identified with the vector space of row vectors (and the row vectors are not just a subspace!)
If we have a nondegenerate inner product $\langle \cdot, \cdot \rangle_1$ on a finite-dimensional vector space V and a nondegenerate inner product $\langle \cdot, \cdot \rangle_2$ on a finite-dimensional vector space W and a linear transformation

$$T : V \to W$$

Then we know (see Definition 5 of the writeup on dual spaces on the webpage!) that there is a dual map

$$T^* : W^* \to V^*$$

given by

$$T^*(\omega)(v) := \omega(Tv).$$

To get the adjoint as in the notes on Prof. Kazdan's webpage, we apply:

$$\varphi^{-1}_{\langle\cdot,\cdot\rangle_1} T^* \circ \varphi_{\langle\cdot,\cdot\rangle_2}$$

(Think about this!)

10. [p. 8 #5] The origin and the vectors X, Y, and $X+Y$ define a parallelogram whose diagonals have length $X+Y$ and $X-Y$. Prove the *parallelogram law*

$$\|X+Y\|^2 + \|X-Y\|^2 = 2\|X\|^2 + 2\|Y\|^2;$$

This states that in a parallelogram, the sum of the squares of the lengths of the diagonals equals the sum of the squares of the four sides.

11. [p. 8 #6] (Math 240 Review)
    a) Find the distance from the straight line $3x - 4y = 10$ to the origin. [It may help to observe that this line is parallel to the plane $3x - 4y = 0$, whose normal vector is clearly $\vec{N} = (3, -4)$.]
    b) Find the distance from the plane $ax + by + cz = d$ to the origin (assume the vector $\vec{N} = (a, b, c) \neq 0$).

12. [p. 8 #8]
    a) If X and Y are real vectors, show that

$$\langle X, Y\rangle = \frac{1}{4}\left(\|X+Y\|^2 - \|X-Y\|^2\right).$$

    This formula is the simplest way to recover properties of the inner product from the norm.
    b) As an application, show that if a square matrix R has the property that it preserves length, so $\|RX\| = \|X\|$ for every vector X, then it preserves the inner product, that is, $\langle RX, RY\rangle = \langle X, Y\rangle$ for all vectors X and Y.

13. [p. 9 #10] (Also done in class)
    a) If a certain matrix C satisfies $\langle X, CY\rangle = 0$ for *all* vectors X and Y, show that $C = 0$.

b) If the matrices A and B satisfy $\langle X, AY\rangle = \langle X, BY\rangle$ for all vectors X and Y, show that $A = B$.

14. [p. 9 #11–12] A matrix A is called *anti-symmetric* (or skew-symmetric) if $A^* = -A$.

    a) Give an example of a 3×3 anti-symmetric matrix.

    b) If A is any anti-symmetric matrix, show that $\langle X, AX\rangle = 0$ for all vectors X.

    c) Say $X(t)$ is a solution of the differential equation $\dfrac{dX}{dt} = AX$, where A is an anti-symmetric matrix. Show that $\|X(t)\| = \text{constant}$. [REMARK: A special case is that $X(t) := \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$ satisfies $X' = AX$ with $A = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ so this problem gives another proof that $\cos^2 t + \sin^2 t = 1$.]

[Last revised: February 19, 2013]