Problem Set 7

DUE: To Shanshan's mailbox, Friday, March 15, 1 pm.

Quadratic Polynomials Using Inner Products

If A is a real symmetric matrix (so it is self-adjoint), then $Q(\vec{x}) := \langle \vec{x},\, A\vec{x}\rangle$ is a quadratic polynomial. Given a quadratic polynomial, it is easy to find the (unique) symmetric symmentic matrix A. Here is an example. Say $Q(\vec{x}) := 3x_1^2 - 8x_1x_2 - 5x_2^2$ To find A, note that $-8x_1x_2 = -4x_1x_2 - 4x_2x_2$ so we can rewite Q as

$$Q(\vec{x}) := 3x_1^2 - 4x_1x_2 - 4x_2x_1 - 5x_2^2.$$

If we let

$$A := \begin{pmatrix} 3 & -4 \\ -4 & -5 \end{pmatrix} \quad \text{[Note } A \text{ is a symmetric matrix]},$$

then it is easy to verify that $Q(\vec{x}) = \langle \vec{x},\, A\vec{x}\rangle$. In the remaining problems we will use this to help work with quadratic polynomials.

1. In each of these find a 3×3 *symmetric* matrix A so that $Q(\vec{x}) = \langle \vec{x},\, A\vec{x}\rangle$.
   a) $Q(\vec{x}) := 3x_1^2 - 8x_1x_2 - 5x_2^2 + x_3^2$.
   b) $Q(\vec{x}) := 3x_1^2 - 8x_1x_2 - 5x_2^2 - x_2x_3 + x_3^2$.
   c) $Q(\vec{x}) := 3x_1^2 - 8x_1x_2 - 5x_2^2 - x_2x_3$.

2. [LOWER ORDER TERMS AND COMPLETING THE SQUARE] Which is simpler:

$$z = x_1^2 + 4x_2^2 - 2x_1 + 4x_2 + 2 \qquad \text{or} \qquad z = y_1^2 + 4y_2^2 \ ?$$

   If we let $y_1 = x_1 - 1$ and $y_2 = x_2 + 1/2$, they are essentially the same. All we did was translate the origin to $(1, -1/2)$.

   The point of this problem is to generalize this to quadratic polynomials in several variables. Let

$$\begin{aligned} Q(\vec{x}) &= \sum a_{ij}x_ix_j + 2\sum b_ix_i + c \\ &= \langle \vec{x},\, A\vec{x}\rangle + 2\langle b,\, \vec{x}\rangle + c \end{aligned}$$

   be a real quadratic polynomial so $\vec{x} = (x_1, \ldots, x_n)$, $\vec{b} = (b_1, \ldots, b_n)$ are real vectors and $A = (a_{ij})$ is a real symmetric $n \times n$ matrix.

   In the case $n = 1$, $Q(x) = ax^2 + 2bx + c$ which is clearly simpler in the special case $b = 0$. In this case, if $a \neq 0$, by completing the square we find

$$Q(x) = a\left(x + b/a\right)^2 + c - 2b^2/a = ay^2 + \gamma,$$

   where we let $y = x - b/a$ and $\gamma = c - b^2/a$. Thus, by translating the origin: $x = y + b/a$ we can eliminate the linear term in the quadatratic polynomial – so it becomes simpler.

a) Similarly, for any dimension n, if A is invertible, using the above as a model, show there is a change of variables $\vec{y} == \vec{x} - \vec{v}$ (this is a translation by the vector $\vec{v}$) so that in the new $\vec{y}$ variables Q hasthe form

$$\hat{Q}(\vec{y}) := Q(\vec{y} + \vec{v}) = \langle \vec{y},\, A\vec{y} \rangle + \gamma \quad \text{that is,} \quad \hat{Q}(\vec{y}) = \sum a_{ij} y_i y_j + \gamma,$$

where γ involves A, b, and c – but no terms that are linear in $\vec{y}$. [In the case $n = 1$, which you should try *first*, this means using a change of variables $y = x - v$ to change the polynomial $ax^2 + 2bx + c$ to the simpler $ay^2 + \gamma$.]

b) As an example, apply this to $Q(\vec{x}) = 2x_1^2 + 2x_1x_2 + 3x_2 - 4$.

3. For $\vec{x} \in \mathbb{R}^n$ let $Q(\vec{x}) := \langle \vec{x},\, A\vec{x} \rangle$, where A is a real symmetric matrix. We say that A is *positive definite* if $Q(\vec{x}) > 0$ for all $\vec{x} \neq 0$, *negative definite* if $Q(\vec{x}) < 0$ for all $\vec{x} \neq 0$, and *indefinite* if $Q(\vec{x}) > 0$ for some $\vec{x}$ but $Q(\vec{x}) < 0$ for some other $\vec{x}$.

a) In the special case $n = 2$ give (simple!) examples of matrices A that are positive definite, negative definite, and indefinite.

b) In the special case where A is an invertible *diagonal* matrix,

$$A = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix},$$

under what conditions is $Q(\vec{x})$ positive definite, negative definite, and indefinite? [REMARK: We will see that the general case can *always* be reduced to this special case where A is diagonal.]

[Last revised: March 9, 2013]