Math 371, Spring 2013, PSet 4

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This problem set will be due Friday, March 22, 2013 at 1 pm in Matti's mailbox. Let k be a field of characteristic $\neq 2, 3$ containing a nontrivial cube root of unity. Consider the field extension $k(\alpha_1, \alpha_2, \alpha_3)/k(a_1, a_2, a_3)$, where $a_1 = \alpha_1+\alpha_2+\alpha_3, a_2 = \alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_1\alpha_3$ and $a_3 = \alpha_1\alpha_2\alpha_3$. We will denote by L the field $k(\alpha_1, \alpha_2, \alpha_3)$ and by K the field $k(a_1, a_2, a_3)$. By problem set 3, L is then an extension of K with $\dim_K L = 6$.

Recall that S_3 has a unique normal subgroup A_3 of order 3 and that S_3/A_3 is of order 2.

1 The Cubic Formula

1. Prove that $Z(k[S_3]) \simeq k \oplus k \oplus k$ (hint: First determine that $\dim_k Z(k[S_3]) = 3$? Once you determine this find three elements $\epsilon_1, \epsilon_2, \epsilon_3$ of $Z(k[S_3])$ such that $\epsilon_i^2 = \epsilon_i, \epsilon_i\epsilon_j = \delta_{ij}\epsilon_i$ and $\epsilon_1 + \epsilon_2 + \epsilon_3 = 1$; prove that this is equivalent to giving the direct sum decomposition). Write out these three elements ϵ_1, ϵ_2 and ϵ_3.

2. Let V be a representation of S_3. Prove that there is a renumbering e_1, e_2, e_3 of the elements you just wrote down so that:

   (a) $e_1 v = v$ if and only if $\gamma v = v$ for all $\gamma \in S_3$.

   (b) $e_2 v = v$ if and only if $\gamma v = v$ for all $\gamma \in A_3$ and, $\gamma v = -v$ for all $\gamma \notin A_3$.

   (c) If $e_3 v = v$ and $\gamma v = v$ for all $\gamma \in A_3$, then $v = 0$.

3. Prove that $M =_{\text{def}} (\epsilon_1 + \epsilon_2)L$ is a field of degree 2 over K. Write down an element σ of K, whose square root $\sqrt{\sigma}$ generates M/K. Write down $\sqrt{\sigma}$ in terms of α_1, α_2 and α_3.

4. Prove that $\mathrm{Aut}(L/M) = A_3$.

5. Find an element τ of M whose cube root $\sqrt[3]{\tau}$ generates K (hint: use e_1 from Problem Set 3, Section 4, Part 1, Number 2, applied to L). Write down $\sqrt[3]{\tau}$ in terms of α_1, α_2 and α_3.

6. Prove that $\{1, \sqrt[3]{\tau}, \left(\sqrt[3]{\tau}\right)^2, \sqrt{\sigma}\left(\sqrt[3]{\tau}\right)^2, \sqrt{\sigma}\left(\sqrt[3]{\tau}\right)^2, \sqrt{\sigma}\}$ form a K-basis of L.
7. Solve for α in terms of this K-basis. Don't forget to use a_1, a_2 and a_3 if you need to!

This is the cubic formula.