Math 371, Spring 2013, PSet 4

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This problem set will be due Friday, April 26, 2013 at 1 pm in Matti's mailbox.

1. Let V be an n-dimensional k-vector space, where k is algebraically closed. Then $\mathrm{End}(V)$ can be identified with the vector space of $n \times n$-matrices with k-coefficients, after choosing a basis B. Let $\mathbf{Funct}(\mathrm{End}(V))$ denote the ring of functions from $\mathrm{End}(V)$ to k, with addition and multiplication given pointwise. Then the basis B gives a ring homomorphism

$$i_B : k[x_{11}, x_{12}, \ldots, x_{(n-1)n}, x_{nn}] \to \mathbf{Funct}(\mathrm{End}(V))$$

given by, if $A : V \to V$ is written as a matrix w.r.t. the basis B as (a_{ij}),

$$i_B(f)(A) = f(a_{11}, a_{12}, \ldots, a_{(n-1)n}, a_{nn}).$$

We denote the image of i_B by $\mathcal{O}_B$.

(a) Prove that i_B is an injective map.

(b) Prove that, if B and B' are two bases, then $\mathcal{O}_B = \mathcal{O}_{B'}$. We then call $\mathcal{O}$ the common image of all the i_B's, the **ring of algebraic functions** on $\mathrm{End}(V)$. By the last problem, $\mathcal{O}$ is isomorphic to a polynomial ring in n^2 variables.

(c) We say that $f \in k[x_{11}, x_{12}, \ldots, x_{(n-1)n}, x_{nn}]$ is an **invariant polynomial** if for all bases B of $V, i_B(f) = i_{B'}(f)$. We denote by $\mathcal{I} \subseteq k[x_{11}, x_{12}, \ldots, x_{(n-1)n}, x_{nn}]$ the collection of invariant polynomials.

   i. Prove that $\mathcal{I}$ is a ring.

   ii. Prove that the coefficients $\beta_1, \ldots, \beta_n$ of the characteristic polynomial $\det(A - tI)$ are elements of $\mathcal{I}$.

   iii. Let $\Delta \subseteq \mathrm{End}(V)$ be the subset of diagonalizable linear transformations with distinct eigenvalues. Prove that a polynomial is invariant when restricted to Δ if and only if it is in the subring T of $k[x_{11}, x_{12}, \ldots, x_{(n-1)n}, x_{nn}]$ generated by $\beta_1, \ldots, \beta_n$.

iv. Prove that $\mathcal{I} = T$.

2. **BONUS QUESTION**: What happens if k is not algebraically closed? What works? What fails?

3. Compute the Galois groups of the splitting fields of the following polynomials over $\mathbb{Q}$:

   (a) $t^4 + t^2 + 1$.

   (b) $t^4 + t^2 + 2$.

   (c) $t^3 - t + 1$.

   (d) $t^4 + 2$ (hint: semi-direct products).

4. Compute the character table of the dihedral group D_5 (we'll go over this in class).

5. Prove that every group of order p^n has nontrivial center.