Representation Theory, Sài Gòn, Summer 2014: Problem Set 1

Aaron Michael Silberstein

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1. Prove the **Peirce decomposition** of an associative algebra: let $e_1, \ldots, e_n$ be elements of a (not necessarily commutative) algebra A with identity 1 such that

   (a) $e_i e_j = \delta_{ij} e_i$ (where δ_{ij} is the delta function, 1 if $i = j$ and 0 otherwise).

   (b) $\sum_i e_i = 1$.

   We call this a **complete, orthonormal collection of idempotents**. Prove that, as an abelian group, $A = \bigoplus_{i,j} e_i A e_j$.

2. Let $\{f_1, \ldots, f_n\}$ be a basis of a finite-dimensional vector space V, and let $\{f_1^*, f_2^*, \ldots, f_n^*\}$ be the dual basis. Prove that the elements

$$v \mapsto f_i^*(v) \cdot f_i \in \mathrm{End}(V)$$

   form a complete, orthonormal collection of idempotents of $\mathrm{End}(V)$. Use the Peirce decomposition to prove that, after choosing a basis of V, every endomorphism of V can be written as a matrix.

3. Let V be a finite-dimensional vector space V over an algebraically closed field k, $\mathrm{End}(V)$ the ring of k-linear maps from V to V, and $\mathrm{Aut}(V)$ the set of invertible elements of $\mathrm{End}(V)$. $\mathrm{Aut}(V)$ acts on the ring $\mathrm{End}(V)$ by conjugation. We say a function $\varphi : \mathrm{End}(V) \to k$ is **algebraic** if it is in the algebra generated by functions of the form $A \mapsto \phi(Av)$ (**monomials**) where $\phi \in V^*$, the dual space of V, and $v \in V$.

   (a) We say that an element of $\mathrm{End}(V)$ is **regular** if it is conjugate to a linear transformation which is diagonal with all eigenspaces of dimension 1. Prove that there is an algebraic function $\varphi : \mathrm{End}(V) \to k$ such that an element A is not regular if and only if it is a zero of φ. Prove that we may take this algebraic function to have integer coefficients with respect to a suitable generating set of monomials (hint: matrix coefficients). Deduce that any algebraic function is determined by its values on the regular elements of $\mathrm{End}(V)$.

(b) Let $\lambda_1, \ldots, \lambda_\ell$ be a finite set of algebraically independent elements of a commutative k-algebra (that is, there are no nontrivial polynomial relations between them with coefficients in k). The **m-th elementary symmetric polynomial** $E_m(\lambda_1, \ldots, \lambda_\ell)$ (for $1 \leq m \leq \ell$) is defined by:

$$\prod_{i=1}^{\ell}(x - \lambda_i) = \sum_{i=0}^{\ell} E_m(\lambda_1, \ldots, \lambda_\ell) x^{\ell - m}.$$

Prove that any symmetric polynomial in the λ_i (any polynomial invariant under changing the indices of i) is a polynomial in the elementary symmetric polynomials.

(c) Let B be a basis of V. Prove that a conjugation-invariant function on $\mathrm{End}(V)$ is determined by its values on elements of $\mathrm{End}(V)$ regular and diagonalizable with respect to B. Prove that the restriction homomorphism of the algebra of conjugation-invariant functions to regular elements of $\mathrm{End}(V)$ diagonalizable on B is an isomorphism onto the symmetric functions of the eigenvalues.