Block TFTs 10/6/8 | 1

Again an Operad $\mathcal{O}$ is a collection of objects $\mathcal{O}(n) \in \mathcal{C}$ where each elt of $\mathcal{O}(n)$ has an action of Σ_n and there exists compositions

$$\mathcal{O}(n) \otimes \mathcal{O}(m) \xrightarrow{i} \mathcal{O}(n+m-1)$$

satisfying

1) equivariance
2) unital
3) Associativity:

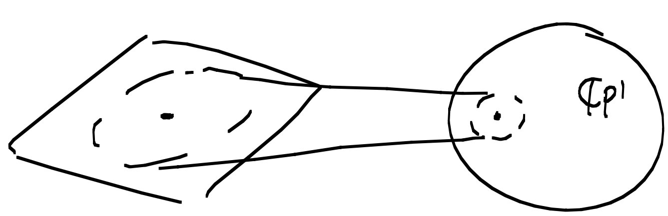
for $f \in \mathcal{O}(a)$, $g \in \mathcal{O}(b)$, $h \in \mathcal{O}(c)$

$(f \circ_j g) \circ_i h = \Big\{$ see last lecture $\Big\}$

Example: $\mathrm{End}(V)(n) = \mathrm{Hom}(V^{\otimes n}, V)$ is an operad.

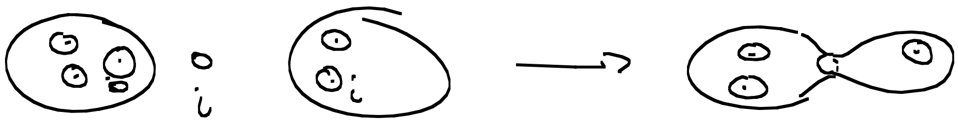
Example: Let $M(n)$ be the Moduli spc. of $\mathbb{CP}^1$'s w/ $(n+1)$ marked pts.

By a marked pt, we mean the image of zero via a holomorphic embedding of the disk in $\mathbb{C}$ to $\mathbb{CP}^1$ so that we get coordinates around each marked pt:



Σ_n acts by permuting pts.

$\circ_i : M(n) \times M(m) \longrightarrow M(n+m-1)$ is performed by gluing the opt to the i^{th} marked pt.



[When we go to higher genus we need the notion of a modular operad]

Plan, first talk about free operads with which construct resolutions of a given one. Get homotopy versions of operads.

2) A_∞-operad is the resolution of Ass

3) Given an operad there is a modular envelope of it $\hat{O}$,

4) $\widehat{(Ass)}_\infty = \bigcup_{g,n} \overline{M}_{g,n}$.

Defn A Σ-module / $\mathcal{C}$ is a sequence of objects $E(n) \in \mathcal{C}$, together w/ Σ_n-actions.

Σ-modules form a category.

$\exists$ forgetful functor : Operads $\xrightarrow{\square}$ Σ-modules

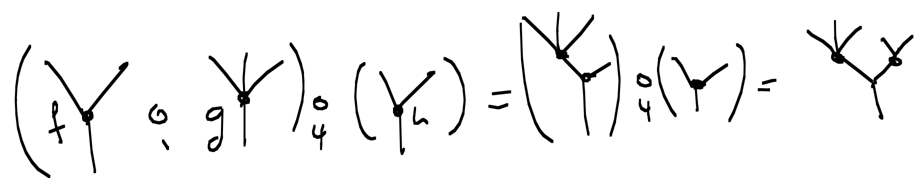
Theorem U has a left adjoint

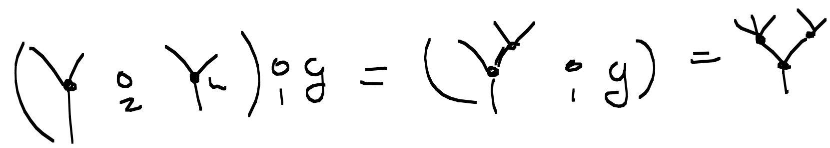
$$F : \Sigma\text{-modules} \longrightarrow \text{Operads}$$ which is called the free operad on E.

The idea of the free operad is that if

$f \in E(2)$, $g \in E(3)$, $h \in E(2)$

$(f \circ_1 g) \circ_4 h = (f \circ_2 h) \circ_1 g$ as trees:





Extend E to all finite sets:

For a finite set S,

Define $E(S) = E(n) \times_{\Sigma_n} \mathrm{Iso}(\langle n \rangle, S)$ } needs to make sense in a symmetric monoidal cat.

has a right $\mathrm{Aut}(S)$-action

For T a tree, set $E(T) = \bigotimes_{v \in V(T)} E(\underbrace{\mathrm{in}}_{\text{set of input edges}}(v))$

Define $\Phi E(n) = \underset{\text{Trees}}{\mathrm{colim}}\, E(T)$ "=" $\bigoplus_{\text{isom. classes of Trees}} E(T)$

$E(T)$ is called an E-decorated tree.

Define $F(E)$ = unitalization of $\Phi(E)$

Defn An ideal of an operad $\mathcal{O}$ is $I = \{I(n)\}$ where

$I(n)$ stable under Σ_n, and

$a \circ_i b \in I(n)$ if one of a or b is in I.

Given an operad $\mathcal{O}$ w/ ideal I,

$\exists\ \mathcal{O}/I = \mathcal{O}(n)/I(n)$.

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note:

$\mathcal{O}$ is an operad in vec. spcs. $\mathcal{O}(1) \otimes \mathcal{O}(1) \to \mathcal{O}(1)$

is an assoc. alg.

Definition a derivation d of an operad $\mathcal{O}$ is a collection

$d_n : \mathcal{O}(n) \to \mathcal{O}(n)$ s.t. for $a \in \mathcal{O}(k)$, $b \in \mathcal{O}(\ell)$

$d_{k+\ell-1}(a \circ_i b) = d_k(a) \circ_i b + (-1)^{|d||a|}(a \circ_i d_\ell(b))$

Let A be an assoc. algebra st. $\dim A_i < \infty\ \forall i$.

Define: cobar construction:

A in degree 0: take $A^*[-1]$

define m: $A^*[-1] \xrightarrow{m^*} A^*[-1] \otimes A^*[-1]$.

$A \otimes A \xrightarrow{m} A$, $m^* : A^* \longrightarrow (A \otimes A)^*$

m^* will extend as a derivation $T(A^*[-1])$

$BA = T(A^*[-1], d)$

<u>Theorem</u> $\underbrace{B}_{\text{cofreely}}BA \simeq A$.

one of these is Bar one is cobar will fix next time.