Block TFTs 10/29/18

recall A_∞-alg.: As formula: $m_k : A^{\otimes k} \longrightarrow A$ sat. a bunch of formulas

via coderivation on $T^c(A[1])$

or an algebra over the planar rooted tree operad.

To define an L_∞-algebra there are 3 ways again:

Formula L graded vec. spc.

$m_k : L^{\otimes k} \longrightarrow L$, $k \geq 1$ of degree $2-k$

satisfying $\sum_{\sigma \in k,l \text{ shuffle}} (-1)^{\sigma} m_{l+1}(m_k(x_{\sigma(1)}, \ldots, x_{\sigma(k)}), x_{\sigma(k+1)} \cdots x_{\sigma(k+l)})$

ex) if $m = 0$ $k \neq 1,2$
Then L is DGLA.

2) coderivation Let V be graded vec. space:
define cofree graded cocommutative coalgebra on V

ΛV = symmetric product on even part
exterior product on odd part

$\Lambda V \subset T^c V$

$$\Delta(v_1 \wedge \cdots \wedge v_n) = \sum_0^n \sum_{k,l \text{ shuffles}} (-1)^{\sigma} (v_{\sigma_1} \wedge \cdots \wedge v_{\sigma_k}) \otimes (v_{\sigma_{k+1}} \wedge \cdots \wedge v_{\sigma_{k+l}})$$

An L_∞-algebra is a coderivation on $\Lambda L[-1]$, w/ $[\delta, \delta] = 0$

Operadic Algebra over rooted non-planar trees operad.
no natural ordering on leaves.

Theorem (Kadeishvili)
[Every A_∞-algebra has a minimal model]

That, $\exists$ an A_∞ structure on $H^*(A, m.)$ s.t. $m_1 = 0$, m_2 induced from A and $\exists$ quasi isom $H^*A \to A$ which is unique up to isom.

ex)

$(\Omega^* M, d)$

We find natural rep. of cohomology classes via Hodge theory.
define Laplacian Δ

$\mathcal{H}^k = \ker \Delta : \Omega^k \to \Omega^k$
$\quad \| $
Harmonic forms

Facts from Hodge Theory:

1) $\Delta \omega = 0 \iff d\omega = 0 = d^* \omega$
2) $(\mathcal{H}^\bullet, 0) \to (\Omega^\bullet, d)$ is quasi-isom.
3) $\mathcal{H}^\bullet$ is fin. dim'l. (Δ elliptic, M cpct.)

let $P : \Omega^\bullet \to \mathcal{H}^\bullet$ projection onto Harmonic forms

$\exists G$ s.t. $\Delta G = G \Delta = 1 - P$ $\qquad$ G "Green's operator"

define $h = d^* G$ $\qquad$ h has degree -1
and $dh + hd = 1 - P$.

problem: $H^\bullet$ is not an algebra (sometimes $H^\bullet$ is an algebra ex if M is a Riemannian symmetric space $M = S^2, \mathbb{C}P^n$, Grass. (no higher Massey Products))

define $m_2(\omega,\eta) = P(\omega\eta)$

Merkulov constructs higher multiplications:

Let (A,d) be DGA, $B \subseteq A$ subcomplex

suppose $\exists h$ of degree -1 s.t. $P = 1 - dh - hd : A \to B$

Then $dP = Pd$. Define $(B, d, \bullet)$

$$x \bullet y = P(xy)$$

Merkulov $\exists$ A_∞-structure on B w/ $m_1 = d$, $m_2 = \bullet$

define $\lambda_n : A^{\otimes n} \to A$ by $\lambda_2(a_1,a_2) = a_1 \cdot a_2$

define $\lambda_1 = -h^{-1}$ (actually h^{-1} dne but λ_1 will only be used in combo $h\lambda_1 = -Id$)

recursively: $$\lambda_n(a_1 \cdots a_n) = \sum_{k+l=n} (-1)^{k+(l-1)(a_1+\cdots+a_k)} h(\lambda_k(a_1 \cdots a_k)) \cdot h(\lambda_l(a_{k+1} \cdots a_{k+l}))$$

lemma $$\sum_{k+l=n+1} \sum_{j=0}^{k-1} (-1)^r \lambda_k(a_1 \cdots a_j, \lambda_l(a_{j+1} \cdots a_{j+l}), a_{j+l+1} \cdots a_n) = 0$$

w/ r = sign.

lemma $$d\lambda_n(a_1 \cdots a_n) + \sum_{j=0}^{n-1} (-1)^{n-1+a_1+\cdots+a_j} \lambda_n(a_1 \cdots, da_{j+1}, a_{j+2} \cdots a_n)$$
$$- \sum_{k+l=n+1} \sum_{j=0}^{k-1} (-1)^r \lambda_k(a_1, \cdots a_j, [d,h]\lambda_l(a_j + \cdots + a_{j+l}))$$

one checks $m_1 = d$, $m_2 = P(\bullet)$, $m_k = P \circ \lambda_k$.

is an A_∞-algebra.

ex) 1) So, if M is cpt. mnfld then $(\mathcal{H}M, 0)$

This A_∞-algebra determines the rational homotopy type of M

(M simply connected).

2) Take $\Omega^{0,\bullet}(X)$ X Kähler mnfld. Then the Kähler

metric gives $\bar\partial^*$, $\bar\partial\bar\partial^* + \bar\partial^*\bar\partial = \Delta_{\bar\partial}$ the $\bar\partial$-laplacian

Hodge identity: $2\Delta_{\bar\partial} = \Delta$ and so kernels coincide.

From this get an A_∞-algebra on the Dolbeault

cohomology $H^{0,\bullet}(X) \cong H^\bullet(X, \mathcal{O}_X)$

More generally, if E is a holomorphic VB w/ Hermitian

metric. Get an A_∞-alg. structure on $H^{0,\bullet}_{\bar\partial}(X, \operatorname{End} E) \cong H^\bullet(X, \underline{\operatorname{End} E})$

holom. endom. sheaf.