Block TQFTs 10/25/18

Topology of aspherical spaces:

X <u>aspherical</u> if $\pi_i(X) = 0$ $i \geq 2$

Homotopy type of X is π_1.

If Γ is a group, $\mathrm{Aut}(\Gamma)$ and $\exists$ ES:

$$1 \longrightarrow Z(\Gamma) \longrightarrow \Gamma \longrightarrow \mathrm{Aut}\,\Gamma \longrightarrow \mathrm{Out}\,\Gamma \longrightarrow 1$$

To what extent is topology determined by π_1?

If X^n is closed aspherical n-mfld, if $X^n \simeq Y^n$

is $X^n \cong Y^n$? (Affirmative: Borel conjecture)

(If actually: $h: X \xrightarrow{\sim} Y^n$ homotopy equivalence, then h homotopic to a homeomorphism)

ex): Known for Γ a cocompact lattice in a Lie grp.

X spc. Let $\mathcal{H}(X)$ = space of homotopy equivalences

$\mathcal{H}$ is an H-space and $\pi_0 \mathcal{H}$ is a grp.

<u>Lemma</u> If X is an aspherical space, then

$$\pi_0 \mathcal{H} X \cong \mathrm{Out}\,\pi_1 X$$

One way to think of B-conjecture: If $f \in \pi_0 \mathcal{H} X$ then $\exists\, \phi$

$\phi \in \mathrm{Homeo}(X)$ w/ $\exists\, \phi \mapsto f$ under $\mathrm{Homeo}(X) \hookrightarrow \mathcal{H} X$

Famous question: If X is an aspherical mfld, if $G \leq \mathrm{Out}\,\pi_1 X$

Assume Borel true for X, each element of G lifts.

Can you lift the group? (often not when G infinite)

One clear obstruction: If you can lift to homeomorphisms, then $\exists$ extension $1 \to \pi_1 X \to \Pi \to G \to 1$ w/ outer action of G on $\pi_1 X$ from the extension is the one given. (Let Π = group gen'd by homeomorphisms of the universal cover which are lifts of elts. of G)

Moduli space of Riemann surface:

equivalent:

1) conformal classes of Riemannian metrics (2-dim'l)
   g_1, g_2 two metrics on same space if they preserve angles: $\exists f$ s.t. $g_1 = e^f g_2$

2) Uniformization Theorem: (Köbe) In each conformal class of metric $\exists$ unique one w/ curvature $0, 1,$ or -1.

3) Algebraic curves (non-singular, 1-dim'l, varieties)

4) cmplx 1-dim'l mflds.

Fix F_g oriented, closed surface of genus g.

A framed Riemann surface is the following data:

$(S, [f])$ $\quad$ S is conformal class of RS

(resp. $g \geq 2$ S RS of curvature -1)

$f : S \to F$ diffeomorphism, $[f]$ its homotopy class.

$(S, [f])$ equiv. to $(S', [f'])$ if $\exists$ a conformal diffeo. (resp. isom)

$h : S \to S'$ st. $f' \circ h \sim f$.

$$\begin{tikzcd} S \arrow[r, "f"] \arrow[d, "h"'] \arrow[dr, phantom, "\circlearrowleft_{ho}"] & F_g \\ S' \arrow[ur, "f'"'] & \end{tikzcd}$$

The set of equiv. classes of framed Riemann surfaces is called <u>Teichmüller space</u> T_g.

Define Γ_g = <u>mapping class group</u> : $\pi_0 \, \mathrm{Diffeo}^+(F)$

= isotopy classes of orientation-preserving diffeo classes of F

= (for Riemann surfaces) $\{$ homotopy classes of diffeos$^+$ of F $\}$

Define action of Γ_g on T_g by

$$\tau \cdot (S, [f]) = (S, [\tau \circ f])$$

$$\boxed{\mathcal{M}_g \equiv T_g / \Gamma_g}$$

T_g will always have fixed pts. coming from surfaces w/ automorphisms.

When $g \geq 2$ there will always be a finite # of autom's.

(autom's means either diffeos or resp. isometries)

slight generalization (add marked pts. to kill automorphisms):

let $s \geq 0$, choose $\{p_1, \dots, p_s\} \in F_g$. an ordered set of pts.

Define $F_g^s \equiv F_g \setminus \{p_i\}$

1) conformal case:

$$(S, (q_1 - q_s), [f]^s) \qquad f : S \longrightarrow F_g \;\big|\; f(q_i) = p_i$$

and $[f]^s$ is a homotopy class rel. $\{p_i\}$ $\Big\}$ defines T_g^s

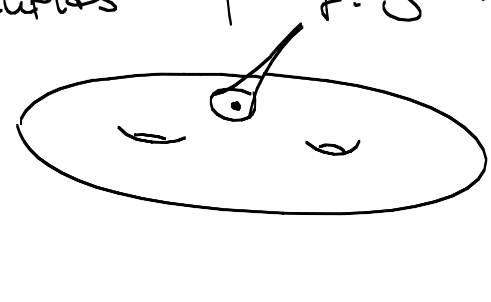
Γ_g^s = homotopy rel. $\{p_i\}$ classes of diffeos.

and then $\mathcal{M}_g^s \equiv T_g^s / \Gamma_g^s$

2) hyperbolic structures:

Complete hyperbolic mflds w/ $f : S \to F_g^s$

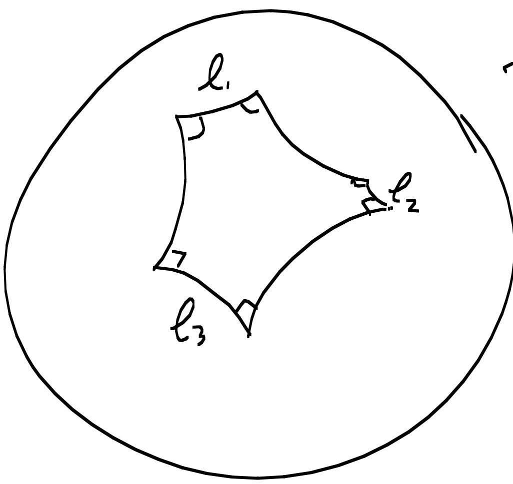
w/ finite area



Coördinates on Teichmuller spc:

(use hyperbolic description):

A hexagon w/ right angles in hyperbolic spc. is determined up to isometry by the lengths of every other side which can be arbitrary:



T1

Note: fact: G finite grp, then $H^*(G;\mathbb{Q}) = \begin{cases} \mathbb{Q} & *=0 \\ 0 & \text{else} \end{cases}$

Mashke's thm

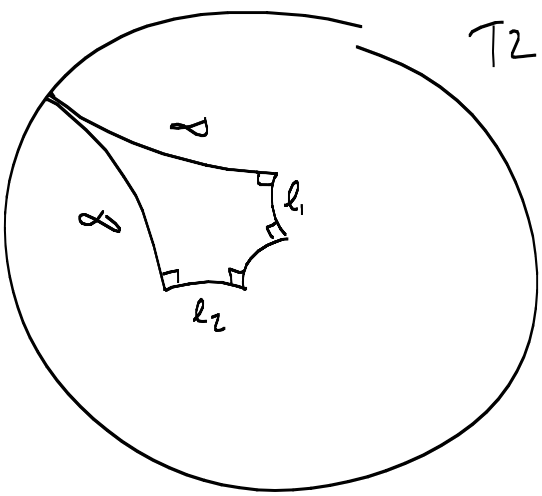
If Γ acts properly and w/ finite stabilizers on X a contractible space, then

$$H^*(X/\Gamma;\mathbb{Q}) \cong H^*(\Gamma,\mathbb{Q})$$

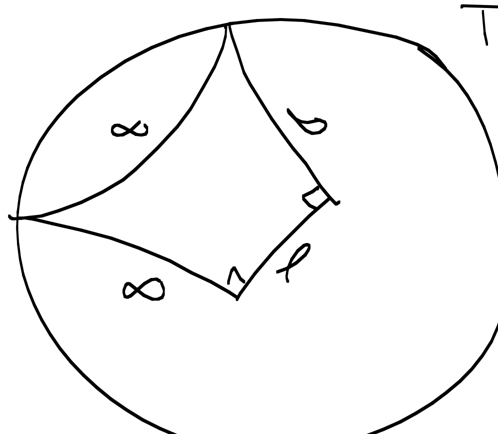
$$\begin{array}{ccc} B\Gamma_x & \to & X \times_\Gamma E\Gamma \\ & & \downarrow \\ & & X/\Gamma \end{array}$$

(not actually a bundle)

Also want to let some & the l_i's be zero or ∞:



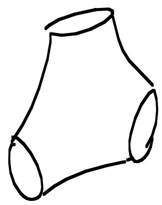
T2



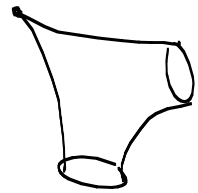
T3

Consider the pair of pants defined by these

T1:



T2:



T3: