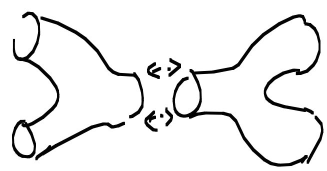
Black TFTs Sept 8 08

Define $n\text{-}\underline{\mathrm{Bord}}_0$ category

$\mathrm{Ob} = \{(n-1)\text{-dim'l, closed, oriented, diff'ble mnflds } \Sigma\}$

$\mathrm{Hom}(\Sigma_0, \Sigma_1) = \{\text{cobordisms starting at } \Sigma_0 \text{ ending at } \Sigma_1\}$

Composition is given by sewing.



Remark: This category has a symmetric monoidal structure:

$\mathcal{C}$ is monoidal if $\exists \otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$,

and $1 \in \mathcal{C}$, nat'l isomorphisms

$$\left\{ \begin{array}{l} (A \otimes B) \otimes C \xrightarrow[\alpha_{ABC}]{\sim} A \otimes (B \otimes C) \\ 1 \otimes A \xRightarrow[\alpha_{1A}]{\sim} A, \quad A \otimes 1 \xRightarrow[\alpha_{A1}]{\sim} A \end{array} \right\}$$

note:

MacLane's Coherence Thm:

As long as $\mathcal{C}, \otimes$ satisfies the coherence:

$$\begin{tikzcd}
(((A\otimes B)\otimes C)\otimes D) \arrow[rr] \arrow[d, "\alpha_{A\otimes B, C, D}"] & & ((A\otimes(B\otimes C))\otimes D) \arrow[d, "\alpha_{A, B\otimes C, D}"] \\
(A\otimes B)\otimes(C\otimes D) \arrow[dr, "\alpha_{A,B,C\otimes D}"'] \arrow[rr, phantom, "\circlearrowright"] & & (A\otimes((B\otimes C)\otimes D)) \arrow[dl, "1_A\otimes\alpha_{B,C,D}"] \\
& (A\otimes(B\otimes(C\otimes D))) &
\end{tikzcd}$$

We have all the identities we want.

So, n-bord_0's symmetric monoidal category structure is given by disjoint union: $\Sigma_1 \otimes \Sigma_2 = \Sigma_1 \amalg \Sigma_2$.

Definition An n-diml TFT is a monoidal functor from $n\text{-bord}_0 \to k\text{-mod}$ (k-commutative)

If Σ^* denotes Σ w/ opposite orientation, then $Z(\Sigma)$ is finite diml. and projective, and $Z(\Sigma^*) \cong Z(\Sigma)$.

Lemma Let k be commutative ring, M, N are two modules together with homoms: $\alpha : k \to M \otimes N$

$\beta : N \otimes M \to k$

satisfying,

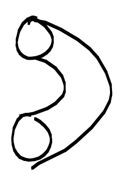
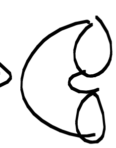
$$\begin{tikzcd} M \arrow[r, "\alpha \otimes 1_M"] \arrow[rr, bend right, "= Id"'] & M \otimes N \otimes M \arrow[r, "1 \otimes \beta"] & M \end{tikzcd}$$

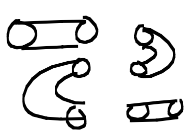
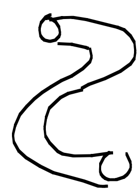
and

$$\begin{tikzcd} N \arrow[r, "1 \otimes \alpha"] \arrow[rr, bend right, "= Id"'] & N \otimes M \otimes N \arrow[r, "\beta \otimes 1"] & N \end{tikzcd},$$

Then, M, N are f.g. and projective.

$X \times I$ is the identity;:

$X \sqcup X^*$  $\xrightarrow[\beta]{} \emptyset$, $\emptyset \xrightarrow[\alpha]{}$  $X \sqcup X^*$

 $=$  $\cong$ cylinder.

$(\alpha \otimes 1) \circ (1 \otimes \beta)$

Definition : A Frobenius algebra (A, τ) is A

an associative algebra (over k a field) w/

$\tau : A \to k$ linear trace and $\exists \langle a, b \rangle \equiv \tau(ab)$

Then $\langle \, , \rangle$ is non-degenerate. (note also $\langle ab, c \rangle = \langle a, bc \rangle$)

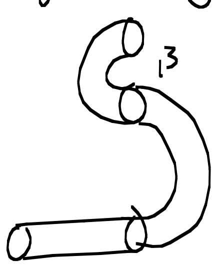
Non degeneracy implies A is finite-dimensional
(Note Frobenius structure can come from some co-algebra conditions)

[**Theorem** A closed 2-d TFT is equivalent to giving a commutative Frobenius algebra.]

pf $Z \Longrightarrow (A, \tau)$:

set $A = Z(S^1)$, structure follows as we saw last time.

(non-degeneracy corresponds to the following picture:)



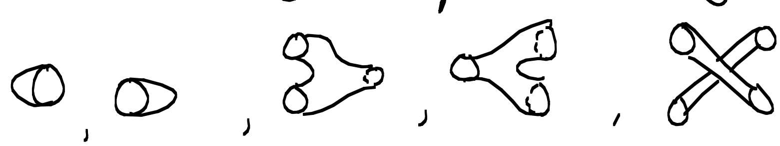
$A \to A$

$\langle , \rangle$ gives $1_A : A \to A$

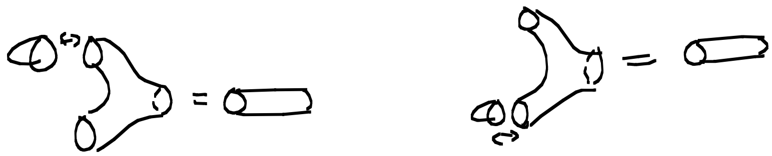
$(A, \tau) \Longrightarrow$ TFT.

Assign $Z(S^1) = A$. $Z(S^1 \amalg \cdots \amalg S^1) = A^{\otimes n}$

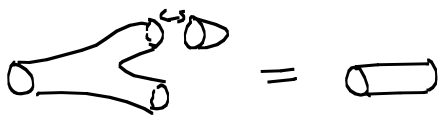
Lemma n-bord is generated by

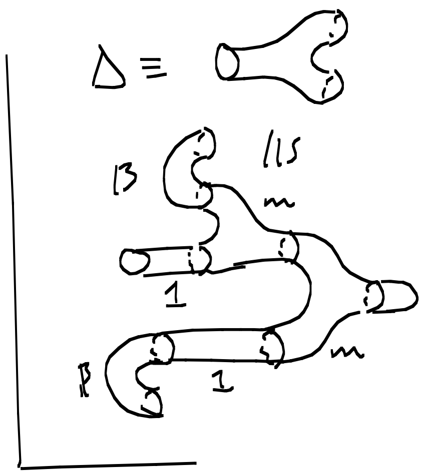


subject to the following relations:

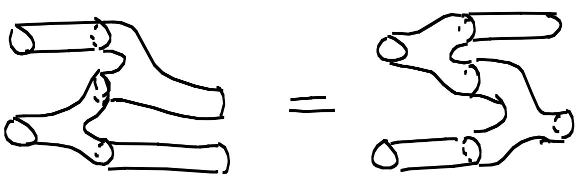


Counit:



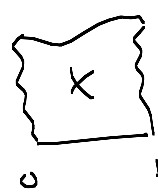


c) (Frobenius Relation)



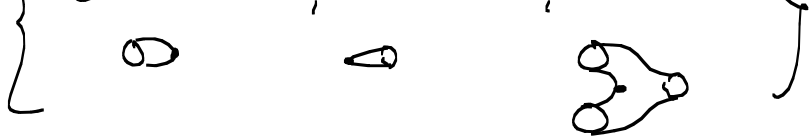
(exercise: write Δ out in (A, τ))

Suppose X is a connected cobordism. Let f:
$f: X \to [0,1]$ from Σ_0 to Σ_1 be a Morse function
s.t. $f(\Sigma_i) = i$.



Passing critical points of f is accomplished by attaching a handle

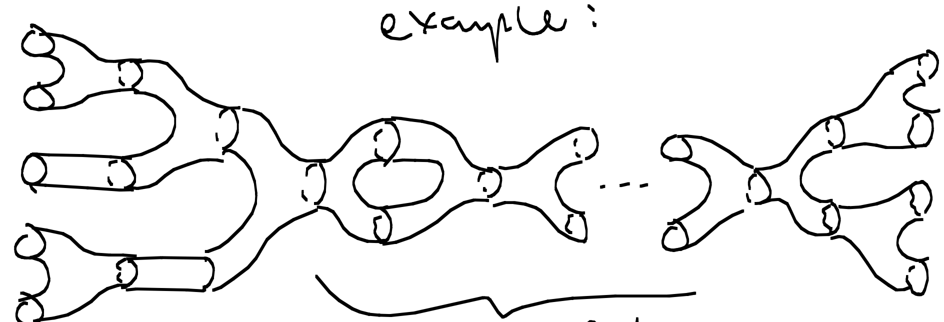
$\{$ 0-handle, 2-handle, 1-handle $\}$



By Morse theory, every cobordism decomposes into generators.

∃ Normal form for connected cobordisms:

example:

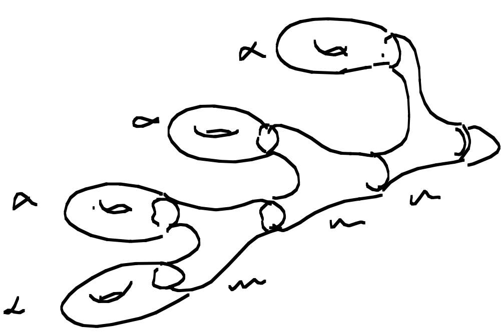


contributes genus

Let (A, τ) Frobenius alg. Then let $\alpha = Z($  $)$ $\in A$

1) $Z(S_g)$ ← closed surface of genus g.

$= \tau(\alpha^g)$:



2) Let $e_i \in A$ be a basis
$e^i \in A$ its dual basis
i.e. $\tau(e_i e^j) = \delta_i^j$
Then $\alpha = \sum e_i e^i$

3) $A \to \mathrm{End}_k(A)$ is injective $\tau(a\alpha) = \mathrm{Tr}(\ell_a)$
$a \to$ left mult by a ... ℓ_a

4) A is semisimple $\iff$ α is invertible

5) If λ_i are the eigenvalues of α, then $Z(S_g) = \sum \lambda_i^{g-1}$

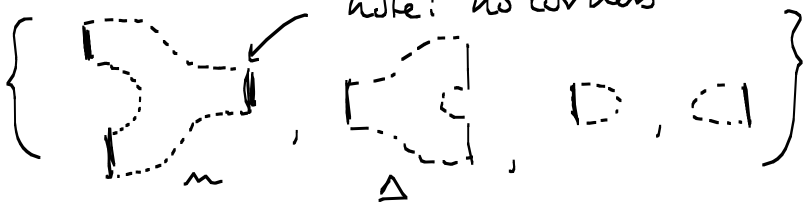
$\underline{\underline{2\text{-Obord}_0}}$ = category of open mnflds w/ bordisms:

Ob = { open, oriented, one manifold

$\mathrm{Hom}(\Sigma_0, \Sigma_1)$ = (open) bordisms

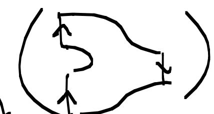
note: no corners

category generated by



A 2-dim'l open TFT is monoidal functor from $2\text{-Obord}_0 \longrightarrow \mathrm{Vect}_k$

Set $A = Z(I)$. We don't get commutativity



<u>Theorem</u> Open 2-d TFTs correspond to noncommutative Frobenius algebra.