Block TFT's 9/18/8

QFT. X mnfld, space-time $\Phi(X)$ = set of fields

Assume $\Phi(X)$ has prob. measure $\mathcal{D}$, . Observable functions on $\Phi(X)$, F, with,

$$\langle F \rangle = \int_{\Phi(X)} F(\varphi)\,\mathcal{D}(\varphi), \quad \langle F_1, F_2 \rangle = \int F_1(\varphi)\, F_2(\varphi)\,\mathcal{D}(\varphi);$$

Where do $\mathcal{D}\varphi$ come from?

In classical field theory, a Lagrangian L on fields, action $S(\varphi) = \int_X L(\varphi, D\varphi, D^2\varphi, \dots)$

Fields of interest are crit. pts. of S. Find them by solving Euler-Lagrange eqns.

Feynman: quantization:

Each field contributes a phase $e^{iS(\varphi)}$ the measure $e^{iS(\varphi)}\mathcal{D}(\varphi)$

$\int e^{t i S(\varphi)} D(\varphi)$ can be calculated asymptotically by stationary phase approx.

[To make this simpler, replace $iS(\varphi)$ w/ $-S(\varphi)$
"Wick rotation"]

Examples: $\Phi(X) = \mathrm{Map}(X, M)$ Let $d\mu$ be a measure on X.

$$\mathcal{D}\varphi = \prod_{x \in X} d\mu(x)$$

so, $F_x(\varphi) = f(\varphi(x))$

$f \in C^\infty(M)$

so $\langle F^1_x(\varphi)\, F^2_x(\varphi) \rangle = 0$.

if $M = L(\varphi)$, then $\prod_{x \in X} e^{-S(\varphi)} d\mu(x) = \mathcal{D}$

then this also has no correlations.

Wiener measure

$$X = [a,b] \subseteq \mathbb{R}, \quad M = \mathbb{R}, \quad \Phi(X) = Map(X, M)$$

($t \in [a,b]$)

$$S\varphi = \frac{1}{2}\int_a^b \dot{\varphi}(t)^2 \, dt$$

Let $F_t(\varphi) = \varphi(t)$. Wiener measure satisfies and is characterized by: 1) F_t is normally distributed $N(0,t)$

$$\langle F_t \rangle_W = 0, \quad \langle F_t^2 \rangle_W = t$$

$$\left(\int_{\Phi(X)} F_t(\varphi) \, \mathcal{D}W(\varphi) =: \langle F_t \rangle_W \right)$$

2) $t \mapsto F_t(\varphi)$ is continuous as in φ.

3) F_t has independent increments!

and $F_t - F_s$ has distribution $N(0, t-s)$

[II means: $F_t - F_s$ is indep. of $t > s$
$F_u - F_v$, $t > s > u > v$,

$$\langle (F_t - F_s)(F_u - F_v) \rangle$$

From these properties it's easy to see:

Let $\underbrace{C(t_1, \cdots, t_n, A_1 \cdots A_n)}_{\text{"cylinder sets"}}$ $\quad a = t_0 < t_1 < t_2 < \cdots < t_n = b$

$A_i \subseteq \mathbb{R}$

$\equiv \{ \varphi \in \Phi \mid \varphi(t_i) \in A_i \}$.

Let $p(t, x) =$ distribution of F_t, $N(0,t)$

$\shortparallel$

$\frac{1}{\sqrt{4\pi t}} e^{-x^2/4t}$ represents the probability that the Wiener walker is in $[x, x+dx]$.

eg, $\mathbb{P}(F_t \in [3,4]) = \int_3^4 \frac{1}{\sqrt{4\pi t}} e^{-x^2/4t} dx$

Define $p(t_1, t_2, x_1, x_2) dx \equiv \mathbb{P}(F_{t_2} = x_2 + dx \mid F_{t_1} = x_1)$

so, $p(t_1, t_2, x_1, x_2) = p(t_2 - t_1, x_2 - x_1)$ by props of WM.

Then, $W(C(t_1 \cdots A_n)) = \int_{A_1} \cdots \int_{A_n} p(t_1 - a, x_1)\, p(t_2 - t_1, x_2 - x_1) \cdots$

$\cdots p(t_n - t_{n-1}, x_n - x_{n-1})$

$dx_1 \cdots dx_n$

So, the properties tell how to define W on
$C(t_1 - A_n)$.

Now plug this premeasure into the Caratheodory machine to get a measure on the whole σ-alg.

(there are problems here: property 2 is probably not satisfied)

$$\prod_{i=1}^{n} \frac{1}{\sqrt{4\pi(t_i - t_{i-1})}} e^{-(x_i - x_{i-1})^2 / 4(t_i - t_{i-1})}$$

If we let mesh of $t_1 < \dots < t_n \longrightarrow 0$ then in the limit (heuristically) this becomes $e^{-\frac{1}{2}\int \dot{\varphi}^2}$

$p(t,x)$ is also the heat kernel: $\left(\begin{array}{c}\text{note: } p \text{ smooth } t>0, \\ p(0,x) = \delta_0\end{array}\right)$

Let $\Phi([a,b]) = Map([a,b], M)$

(k is either 1 or ½ depending on conventions used)

Let M be cpct Riemannian Mfld, $L^2(M, dVol)$ exists
$$\begin{array}{c} L^2(M, dVol) \\ \parallel \\ \mathcal{H}_M \end{array}$$

$\frac{\partial}{\partial t} p(t,x) + \Delta p(t,x) = 0$ heat equ. $p(t,x)$ is also the prob. density function for $f(F_t(x))$

(self-adjoint)

Let $\mathcal{U}_t = e^{-t\Delta}$ is the operator defined by (smoothing operator)

$$\underbrace{(\mathcal{U}_t G)}_{\text{heat semigroup}}(m) = \int_M p(t, m, n)\, d\mathrm{vol}(n)$$

Let $\Omega = 1 \in L^2(M)$,

Theorem

If $a < t_1 < \cdots < t_n = b$ $\quad f_1 \cdots f_n \in C^\infty(M)$

Then $\langle f_n(\varphi(t_n)) f_{n-1}(\varphi(t_{n-1})) \cdots f_1(\varphi(t_1)) \rangle_w$

$$= \langle \Omega, \mathcal{U}_{b-t_n} f_n \mathcal{U}_{t_n - t_{n-1}} f_{n-1} \mathcal{U}_{t_{n-1} - t_{n-2}} \cdots f_1 \mathcal{U}_{t_1 - a} \Omega \rangle$$

can form $V_t = e^{i\Delta t}$ unitary