Block TFTs 9/22/18

QFT: X spc-time. $\Phi(X)$ a space of fields

classically, an action $S(\varphi)$. Quantumly,

$$\int_{\Phi(X)} e^{iS(\varphi)}\mathcal{D}(\varphi) = Z(X).$$

Wick rotation:

$$\int_{\Phi(X)} e^{-S(\varphi)}\mathcal{D}(\varphi)$$

For string theory, think of X as the world-sheet of some string.

Want to understand the interface btwn the gluing of two world-sheets.

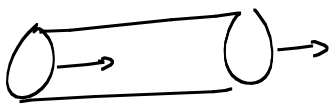
Assume also assign spaces fields to ∂X:

$$\Phi(\partial X) \qquad \Phi(X) \to \Phi(\partial X)$$

For $\alpha \in \Phi(\partial X)$ $\qquad \Phi(X,\alpha) = \{\varphi \in \Phi(X) \mid \varphi|_{\partial X} = \alpha\}$

Define $\quad Z(X)(\alpha) = \displaystyle\int_{\Phi(X,\alpha)} e^{-S(\varphi)}\mathcal{D}(\varphi)$

We want to assume ∂X is oriented : the orientation of the normal represents the "arrow of time" :



If X together $\partial X = Y_0^* \amalg Y_1$, with time pointing from Y_0 to Y_1. We have the following picture:

$$\begin{tikzcd}
 & \Phi(X) \arrow[dl, "p_0"'] \arrow[dr, "p_1"] & \\
\Phi(Y_0) & & \Phi(Y_1)
\end{tikzcd}$$

<u>locality of fields</u>

gluing

$$\begin{tikzcd}
 & & \Phi(X' \circ X) \arrow[dl, "r_0"'] \arrow[dr, "r_1"] & & \\
 & \Phi(X) \arrow[dl] \arrow[dr] & & \Phi(X') \arrow[dl] \arrow[dr] & \\
\Phi(Y_0) & & \Phi(Y_1) & & \Phi(Y_2)
\end{tikzcd}$$

restriction

is a pullback. <u>locality of S</u> is the fact :

$$S_{X' \circ X} = S_{X'} \circ r_1 + S_X \circ r_0$$

Assume to each Y (a bdry component) a Hilbert space $\mathcal{H}_Y = L^2(\Phi(Y); \mathcal{D}_Y)$ (states)

To $X : Y_0 \rightsquigarrow Y_1$, define: $Z(X) : \mathcal{H}_{Y_0} \to \mathcal{H}_{Y_1}$ by

$$Z(X) = p_{1*} e^{-S_X} p_0^*$$

$$\left[\begin{tikzcd} & \Phi(X) \arrow[dl, "p_0"'] \arrow[dr, "p_1"] & \\ \Phi(Y_0) & & \Phi(Y_1) \end{tikzcd}\right]$$

we want to have

$$\langle g, Z(X) f \rangle_{H_{Y_1}} = \int_{\Phi(X)} \bar{g}(p_1(\varphi)) f(p_0(\varphi)) e^{-S_X(\varphi)} \mathcal{D}\varphi$$

note:
If $\partial X = \phi$, then $Z(X) = \int e^{-S(\varphi)} \mathcal{D}\varphi$.

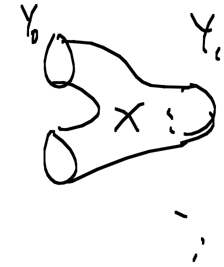
If $X' \circ X$, then

$Z(X' \cdot X) = Z(X') \circ Z(X)$ (follows formally by locality of fields and locality of S plus gluing of measures)

From all this, you get that to each Y, a vec. spc. Z_Y, to each $X : Y_0 \to Y_1$, a map

$Z(X) : Z_{Y_0} \to Z_{Y_1}$, satisfying the gluing axioms and the other properties of a TFT.

Suppose that on 2-Bord_0 we also include a volume form on each cobordism. (ω).

ex) 

$\Phi(Y) \equiv \{C^\infty \text{ funcs on } Y\}$

$Z_{Y_i} = L^2(\Phi(Y), \mathcal{D}_Y)$

Theorem (Moser)

If (M, ω_1), (M, ω_2) are two volume forms on M, then $\exists$ a diffeo: $f: M \to M$ s.t. $f^*\omega_1 = \omega_2$

So, the only new data to 2-Bord is a number

$2\text{-Bord} \times (0, \infty)$. (Cartesian product on morphisms)

Theorem 2-TVFT's (volume) are in 1to1 corresp. w/ the following data:

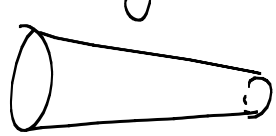
$\{$ A a topological algebra w/ a non-degenerate trace $\tau: A \to \mathbb{C}$ $\}$

$U_t : A \to A$ a nuclear mapping (think trace-class) w/

(think $e^{-t\Delta}$)

$U_t U_s = U_{t+s}$

Operator is defined by the cobordism:



the volume-decreaser

$u_t \to 1$ as $t \to 0$. , also $\exists\ \varepsilon_t$'s.

ε_t:  disks of volume t

$\varepsilon_t \to 1_A$ as $t \to 0$. (an approximate identity)

Next step: include in morphisms $Y_0 \to Y_1$,

i) complex analytic types btwn Y_0, Y_1,

ii) conformal types

iii) const. curvature types

Get moduli spaces $M(g,n)$, compactify them, $\overline{M(g,n)}$, find a combinatorial model for $\overline{M(g,n)}$.