Blocke TFT's 9/26/08

recall: 2-d OCTFT is a $\otimes$-functor

$$2\mathrm{OCBord}_0^1 \longrightarrow \mathrm{Vect}_{/k}$$

$$\| \\ \left\{ \begin{array}{l} \text{1-mnfld} \\ + \text{ cobordisms} \\ \text{up to diffeo} \end{array} \right\}$$

2OCBord parametrized by 1-mnfds and conformal classes of cobordism.

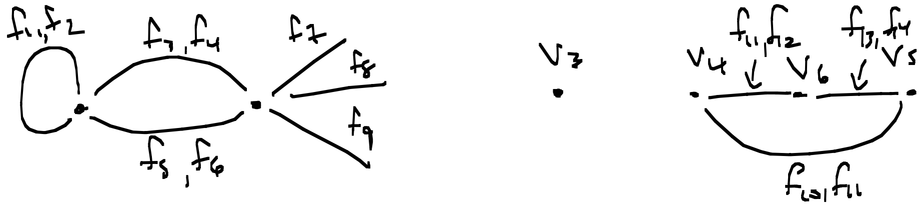
2OCBord(Σ_1, Σ_2) will consist of conformal classes of cobordisms but together the set of these will form a top. spc. (so we get a top. category).

Understanding these will involve a lot of effort.
We will study operads for a couple lectures:

<u>Definition</u> A graph Γ consists of

1) A finite set of vertices $V(\Gamma)$
2) A finite set of flags $F(\Gamma)$
3) $p : F(\Gamma) \longrightarrow V(\Gamma)$
4) an involution $\sigma : F(\Gamma) \to F(\Gamma)$

For $v \in V(\Gamma)$, let $p^{-1}(v)$ be all flags emanating from v

ex) 

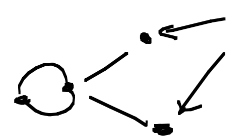
→ define the legs of a graph $L(\Gamma)$ are the fixed pts. of the involution.

→ the edges of Γ are the free orbits of σ.

→ a flag $f \in F(\Gamma)$ if not fixed determines an orientation on the corresponding edge

A graph given as above has a geometric realization $|\Gamma|$ whose 0-cells are $V(\Gamma)$, $\underbrace{L(\Gamma)}$

on these we are adding the endpt:

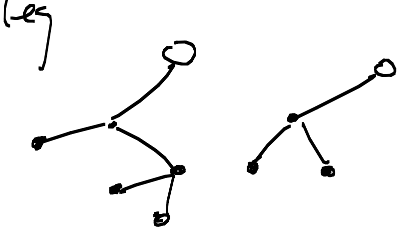


whose 1-cells are $E(\Gamma), L(\Gamma)$

$\pi_0 \Gamma$ = set of components

A forest is a graph Γ whose connected components are contractible and all vertices are at least trivalent.

A rooted forest is a forest for which each connected component we've chosen a leg

ex) 

An isom. of these objects is defined in the obvious way.

Define some categories:

$\mathcal{G}$: $\quad Ob = \{$ a pair of finite sets and a map between them $|I \longrightarrow J|\}$

$$Mor\,(I \to J,\ K \to L) = \left\{ \text{graphs } \Gamma : \begin{array}{c} |F(\Gamma) \xrightarrow{p} V(\Gamma)| \\ \downarrow \\ |L(\Gamma) \to \pi_0(\Gamma)| \end{array} \right\}$$

composition:

let $\Gamma_1 : |F(\Gamma_1) \to V(\Gamma_1)| \to |L(\Gamma_1) \to \pi_0\Gamma_1|$

$\Gamma_2 : |F(\Gamma_2) \to V(\Gamma_2)| \to |L(\Gamma_2) \to \pi_0\Gamma_2|$

assume $\exists$ isom

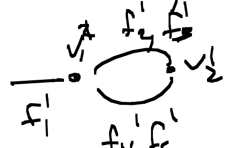
$$|L(\Gamma_2) \to \pi_0\Gamma_2| \to |F(\Gamma_1) \to V(\Gamma_1)|$$

isom

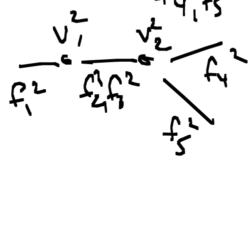
isom

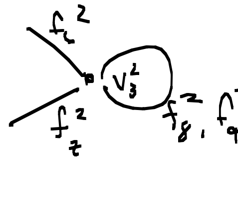
Compose $\Gamma_1 \circ \Gamma_2$ by inserting Γ_2 into Γ_1:

i.e. each vertex $v \in V(\Gamma_1)$ gets replaced by the component of Γ_2 using the above identification and the flags are glued at v to the corresp. legs.

ex) Γ_1 

: $|\{f_1, \dots, f_5\} \to \{v_1, v_2\}|$

Γ_2 

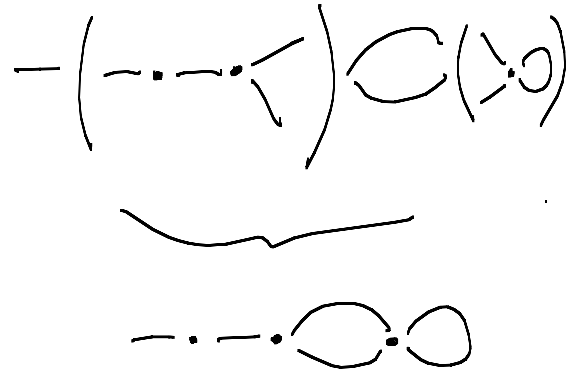


: $|\{f_1^2 - f_9^2\} \to \{v_1^2\ v_2^2\ v_3^2\}|$

So, $\exists$ an isom $\left\{ \begin{array}{ll} \varphi_1 & \pi_0 \Gamma_2 \xrightarrow{\sim} V(\Gamma^1) \\ \varphi_2 & L(\Gamma_2) \xrightarrow{\sim} F(\Gamma_1) \end{array} \right\}$

$$\left\{ \begin{array}{ccc} f_1^2 & & f_1^1 \\ f_4^2 & \xrightarrow{\varphi_2} & f_2^1 \\ f_5^2 & & f_4^1 \\ f_6^2 & & f_3^1 \\ f_2^2 & & f_5^1 \end{array} \right\}$$

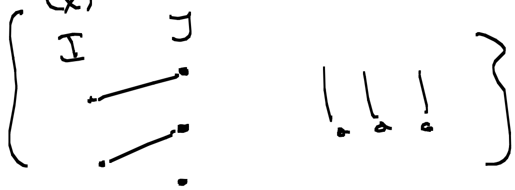
So,



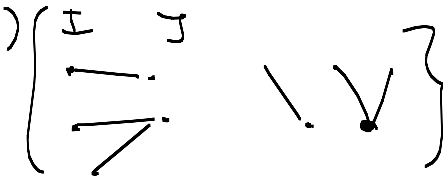
If $|I \to J|$ is an object, $1_{|I\to J|}$ is the graph

w/ $L(\Gamma) = F(\Gamma) = I$
$V(\Gamma) = \pi_0 \Gamma = J$

ex)

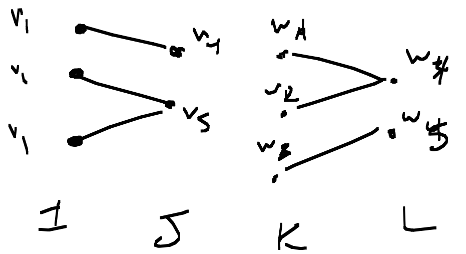


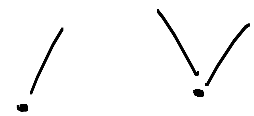
ex)



and if $|I \to J| \to |K \to L|$ is an isomorphism of pairs of sets and morphisms then we get an isom. in our category.

$F(\Gamma) = I$, $V(\Gamma) = J$, $L(\Gamma) = K$, $\pi_0 \Gamma = L$



$\Longrightarrow \Gamma =$  is an isom.

This is symmetric monoidal:

on objects: $|I \to J| \otimes |K \to L| = |I \sqcup K \to J \sqcup L|$

on morphisms: $\Gamma_1 \otimes \Gamma_2 = \Gamma_1 \sqcup \Gamma_2$

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2 subcategories.

$\mathcal{F}$: <u>forest</u> <u>category</u> $\quad$ $Ob\,\mathcal{F} = |I \to J|$ where all fibers of the map have at least 3 pts.

$Mor\,\mathcal{F}$ = Forest graphs.

$\mathcal{F}_r$: <u>rooted</u> <u>forests</u> $\quad$ $Ob\,\mathcal{F}_r = |I \to J|$ as above

$Mor\,\mathcal{F}_r$ = rooted Forest graphs.

These are symmetric, monoidal subcategories.

<u>Definition</u> Let $\mathcal{C}$ be a symmetric monoidal category. An <u>operad</u> in $\mathcal{C}$ is a $\otimes$ functor from $\mathcal{F}_r \longrightarrow \mathcal{C}$

a <u>cyclic operad</u> $\mathcal{F} \to \mathcal{C}$

a <u>modular</u> <u>operad</u> $\mathcal{G} \to \mathcal{C}$