| Classical GW thy 2 | Givental || 7/21/08

$\langle \phi_1 \psi_1^{k_1}, \ldots, \phi_N \psi_N^{k_N} \rangle$

"generalized GW invariants" - my phrase

univ. fibration
$X_{g,n+1,d}$
$\downarrow ft_{n+1}$
$X_{g,n,d}$

3 eqns.: string, dilaton, divisor

ϕ_α basis of in $H^*(X)$

$$\left\{ \begin{array}{ll} \underline{\text{string}}: & (ft_{n+1})_* 1 = 0 \\ \underline{\text{dilaton}}: & (ft_{n+1})_* \psi_{n+1} = 2g-2+n \\ \underline{\text{divisor}}: & (ft_{n+1})_* (ev_{n+1}^* p) = \int_{[d]} p \\ & (p \in H^2(X)) \end{array} \right\}$$

Want: $\langle 1, t(\psi), \ldots, t(\psi) \rangle_{g,n+1,d}$

w/ $t(\psi) = \sum_{\alpha, k > 0} \phi_\alpha \psi^k t_k^\alpha$

$\{\psi_i\}$ are $c_1(L_i)$

$\tilde{\psi}_i$ are $ft_{n+1}^*(\psi_i)$

if $\langle 1, t(\tilde{\psi}), \ldots, t(\tilde{\psi}) \rangle_{g,n+1,d} = 0$

but not if we use ψ. What is the difference?

$\psi_i - \tilde{\psi}_i$? fibers of ψ_i and $\tilde{\psi}_i$ are identified except when the $n+1^{st}$ marked pt. meets the i^{th} one.

$\psi_i - \tilde{\psi}_i = PD(\sigma_i)$. $\tilde{\psi}^k = (\psi - \sigma_i)^k$

$= \psi^k \pm \sigma_i^k$

$= \psi^k - (t_i^{k-1})(\sigma_i)$

so, $0 = \langle 1, t(\tilde{\psi}), \ldots, t(\tilde{\psi}) \rangle_{g,n+1,d} = \langle 1, t(\psi), \ldots, t(\psi) \rangle_{g,n+1,d} -$

$- 2 \langle t(\psi), \ldots, t(\psi) \rangle_{g,n,d}$

string equ: $\left[\langle 1, t(\psi), \ldots, t(\psi) \rangle_{g,n+1,d} = 2 \langle \frac{t(\psi) - t(0)}{\psi}, t(\psi), \ldots, t(\psi) \rangle_{g,n,d} \right]$

(the other terms cancel in the product) since $\psi_i \sigma_i$ is chern class of trivial bundle

$$\left[\langle t(\psi), t(\psi), \ldots, \psi \rangle_{g,n+1,d} = 2g-2+n \langle t(\psi), \ldots, t(\psi) \rangle_{g,n,d} \right.$$

dilaton equ.

$$\left[\begin{aligned} \langle p, t(x), \ldots, t(x)\rangle_{g,n+1,d} &= \left(\int_d p\right)\langle t(x), \ldots, t(x)\rangle_{g,n,d} \\ &+ n\left\langle p\,\frac{t(x)-t(0)}{x}, t(x), \ldots, t(x)\right\rangle_{g,n,d} \end{aligned}\right]$$

divisor eqn.

Introduce $\mathcal{F}_g \equiv \sum_{d,n} \frac{Q^d}{n!}\langle t(x), \ldots, t(x)\rangle_{g,n,d}$

$Q^d = Q_1^{d_1} \cdots Q_r^{d_r}$ where $r = \mathrm{rk}\,(H_2(X,\mathbb{Z}))$

$P_1 - P_r \qquad d_i = \int_d P_i$

$\underbrace{\mathcal{Q}}_{\text{"Curly }Q\text{"}} \equiv \underbrace{\mathbb{Q}}_{\text{Rat'l \#'s}}[\![\underbrace{Q_1 \cdots Q_r}_{\text{formal variables.}}]\!]$ = Novikov's Ring

$\mathcal{F}_g$ - called "genus-g descendent potential"

$\partial_1 \mathcal{F}_g \qquad \{\phi_\alpha\},\ \partial = \frac{\partial}{\partial t_0^{\sim}}$

$t(x) = \underline{t_0} + \underline{t_1}x + \cdots$

$$\frac{\partial}{\partial t_0}\left(\mathcal{F}_g\right) = \sum \frac{Q^d}{n!}\langle 1, t(x), -, t(x)\rangle_{g,n+1,d}$$

$$\overset{\text{string eqn.}}{=} \sum \frac{Q^d}{n!}\langle \underbrace{t(x)-t(0)}_{4}, \underbrace{t(x), -, t(x)}_{n}\rangle_{g,n+1,d}$$

$\left(\partial_1 \text{ means } \frac{\partial}{\partial t_0^1}\right)$

$$\boxed{\partial_1 \mathcal{F}_g = \sum t_k^\alpha \partial_{t_{k-1}^\alpha} \mathcal{F}_g}$$

Poincaré Pairing

in genus $=0$, $\langle 1, t(x), t(x)\rangle_{0,3,0} = \int_X (\ldots) = \frac{1}{2}(t_0, t_0)$

$$\left\{\begin{array}{l} X_{0,3,0} = X \times \overline{\mathcal{M}}_{0,3} = X \\ X_{1,1,0} = X \times \overline{\mathcal{M}}_{1,1} \end{array}\right.$$

i.e. $(a,b) = \int_X a\cup b$

$$X_{1,1,0} = (\underset{\dim X}{X} \times \overline{\mathcal{M}}_{1,1} \quad ?)$$

Dilaton Eqn.

$$\boxed{\begin{aligned}\partial_{p_i}\mathcal{F}_g &= Q_i\frac{\partial}{\partial Q_i}\mathcal{F}_g + \sum_{\alpha,\beta} t_k^\alpha p_{i\alpha}^\beta \frac{\partial}{\partial t_{k-1}^\beta}\mathcal{F}_g \\ &+ \delta_{g,0}\frac{(t_0, p t_0)}{2} + \delta_{g,1}\left(\frac{2}{1}\right)\end{aligned}}$$

$\{p_i\}$ basis for $H_2(X,\mathbb{Z})$

Divisor Equ :

$\langle x_1, \dots \rangle \qquad 2g - 2 + n$

$$\frac{\partial}{\partial t_i} F_g = \sum t_k^\alpha \partial_{t_k^\alpha} F_g + (2g-2) F_g + \cancel{\delta_{g,0}}^{\,0} + \delta_{g,1}$$

correction term

$$t_0 + \underline{t_1}\, x + t_2 x^2$$

$$\delta_{g,1} : \langle x \rangle_{1,1,0} = \frac{\chi(X)}{24}$$

$$\int_{X \times \overline{M}_{1,1}} \text{Euler}\left(\text{"} \underbrace{H^1(\Sigma, \mathcal{O}_\Sigma)}_{\text{"Hodge" Bundle}} \otimes TX \text{"} \right)$$

trivial bundle of curve

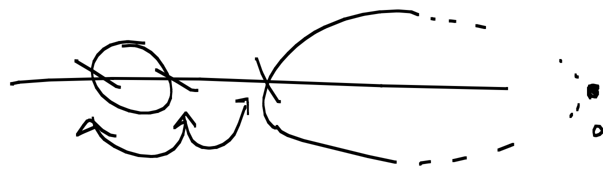
$$||$$

$$H^*(\Sigma, f^* TX), \; H^*(\Sigma, TX)$$

$$\int_{X \times \overline{M}_{1,1}} \text{Euler}\left(TX \otimes \text{Hodge bundle } \mathbb{H}\right) \psi = \int_X \text{Euler}(TX) \cdot \int_{\overline{M}_{1,1}} \psi$$

$$= \chi(X) \cdot \int_{\overline{M}_{1,1}} \psi$$

Thicke of $M_{1,1} = \overline{M}_{0,4}/S_3$ ← permute 3 pts of intersection w/ axes :



S_3-action.

$$\langle \chi, 1,1,1 \rangle_{0,4} = (\underbrace{\cancel{2g} - 2 + n}_{+1}) \overset{6}{} \langle 1,1,1 \rangle_{0,3,0} \langle \chi, t(\chi), t(\chi) \rangle_{0,3,0}$$

$$= \frac{1}{6 \cdot 2} \cdot \frac{1}{2} = \frac{1}{24}$$

↑ 2-fold symmetry of elliptic curves

so result of correction term is $\frac{\chi(X)}{24}$

$$\boxed{\frac{\partial}{\partial t_1^1} F_g = \left(\sum t_1^\alpha \partial_{t_k^\alpha} F_g\right) + (2g-2) F_g + \frac{\chi(X)}{24} \delta_{g,1}}$$

Dilaton Eqn.