DM spaces + KdV Hierarchy 2 | Kai Cieliebak

KdV equ.: $\boxed{u_t = u u_x + \frac{1}{12} u_{xxx}}$ $u(x,t)$

$u(x,0) = u_0(x)$. ICs

$\{\,,\,\}$ Poisson Bracket on mnfld P

(Bracket on $C^\infty(P)$ which is a <u>derivation</u> and is a <u>Lie bracket</u>)

$$H \rightsquigarrow X_H f = \{H, f\} \rightsquigarrow \dot{x} = X_H(x) \quad \frac{d}{dt} f(x(t)) = \{H, f\}.$$

$$\{\,,\,\} \leftrightarrow B \in \Omega_2(P)$$

$$B_x(d_x f, d_x g) = \{f, g\}(x).$$

Jacobi id. for $\{\,,\,\} \longleftrightarrow [B,B] = 0.$ Schouten Bracket

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A bi Hamiltonian structure is a 2-dim'l space of Poisson structures: $a_1 \{\,,\,\}_1 + a_2 \{\,,\,\}_2$ $a_1, a_2 \in \mathbb{R}$

Poisson Brackets corresp to B_1, B_2 and

$[B_1, B_2] = 0.$

Lemma Suppose: $H_0, H_1, \ldots$ satisfy

$$\{\,\cdot\,, H_{p+1}\}_1 = \{\,\cdot\,, H_p\}_2 \quad , \; p = 0, 1, \ldots$$

then $\{H_p, H_q\}_1 = \{H_q, H_p\}_2 = 0 \quad \forall\, p, q$

Pf suppose $q = p + 2m$ (or $q = p + 2m + 1$ case similar)

$$\{H_p, H_q\}_1 = \{H_p, H_{q-1}\}_2 = -\{H_{q-1}, H_p\}_2 = -\{H_{q-1}, H_{p+1}\}_1$$

$$= \{H_{p+1}, H_{q-1}\}_1 = \cdots\cdots$$

$$= \{H_{p+m}, \underbrace{H_{q-m}}_{p+m}\} = 0 \quad \Rightarrow \text{all zero} \quad \blacksquare$$

The method then involves start w/ some H_0

then define $\{\,\cdot\,, H_{p+1}\} = \{\,\cdot\,, H_p\}$

View the KdV eqn as an ODE on some loop spc.:

$P = C^\infty(S^1, \mathbb{R})$ take $u \in P$

$T_u P \cong P$ since P linear spc.

$\cong T_u^* P$ via pairing $\int_{S^1} f(x) g(x)\, dx$,

$H(u) = \int h(x, u, \underbrace{u', u'', \dots}_{\text{any finitely many derivatives}})\, dx \qquad H : P \to \mathbb{R}$

$$d_u H \cdot f = \int \left[\frac{\partial h}{\partial u} f + \frac{\partial h}{\partial u'} f' + \frac{\partial h}{\partial u''} f'' + \dots\right]$$

$$= \int \underbrace{\left(\frac{\partial h}{\partial u} - \left(\frac{\partial h}{\partial u'}\right)' + \left(\frac{\partial h}{\partial u''}\right)'' \mp \dots\right)}_{d_u H} f$$

Consider : $\quad h_0 = \frac{u^2}{2} + \frac{\varepsilon^2}{12} u''$

{recall that $T_u^* P \subseteq P$ } $\quad h_1 = \frac{u^3}{6} + \frac{\varepsilon^2}{24}(u'^2 + 2uu') + \frac{\varepsilon^2}{240} u^{(4)}$

$\to B_u^1(f,g) = \int f' g$

$B_u^2(f,g) = \int \left(u' f + 2u f' + \frac{\varepsilon^2}{4} f''\right) g =$

$= \int \left(u(f'g - fg') + \frac{\varepsilon^2}{4} f''' g\right).$

Lemma $q_1\{\,,\,\}_1 + q_2\{\,,\,\}_2$ is bihamiltonian structure

Hamiltonian flows $\delta_x(u) = u(x)$ $\quad d_u\delta_x = \delta_x$

$$\{H,\delta_x\}_1(u) = D'_u(d_uH, \delta_x) = \int (d_uH)'\delta_x\,dx$$

$$= \frac{d}{dx}(d_uH)(x)$$

$$u_t = \underbrace{\{H,\cdot\}_1}_{= X^1_H} \iff u_t(x) = \frac{d}{dx}(d_uH)(x) = \frac{d}{dx}\left(\frac{\partial h}{\partial u} - \left(\frac{\partial h}{\partial u'}\right)' + \dots\right)$$

$$d_uH_1 = \frac{u^2}{2} \cancel{+\frac{\varepsilon^2}{12}u''} - \cancel{\left(\frac{\varepsilon^2}{12}u'\right)'} - \left(\frac{\varepsilon^2}{12}u\right)''$$

$$= \frac{u^2}{2} + \frac{\varepsilon^2}{12}u''$$

$$u_t = X^1_{H_1}(u) \iff u_t = uu' + \frac{\varepsilon^2}{12}u'''$$

$$\iff u_t \overset{!}{=} \frac{1}{3}X^2_{H_0}(u)$$

$\Rightarrow \exists$ infinite family of commuting Hamiltonians

$H_0, H_1, \cdots$ defined by (actually there is ambiguity here)

$$\{H_n, \cdot\}_1 = \frac{1}{2n+1}\{H_{n-1}, \cdot\}_2$$

$$\Rightarrow \left[\frac{d}{dx}(d_u H_n) = \frac{1}{2n+1}\left(u' + 2u\frac{\partial}{\partial x} + \frac{\varepsilon^2}{4}\frac{\partial^3}{\partial x^3}\right)(d_u H_{n-1})\right]$$

Define $u(x = t_0, t = t_1, t_2, t_3, \cdots)$ by

$$\boxed{\frac{\partial u}{\partial t_n} = X^1_{H_n} = \frac{\partial}{\partial t_0}(d_u H_n)}$$

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u will be a soln of KdV if $t_2 \cdots t_N$ is fixed.
(since h_1 generates the KdV eqn.)

$\boxed{\text{Witten's Conjecture}}$

Interested in $\displaystyle\int_{\overline{M}_{g,n}} \psi_1^{d_1} \cdots \psi_n^{d_n} =$

Witten notation

$= \langle \tau_{d_1} \cdots \tau_{d_n} \rangle_{g,n}$

(generating function)

Introduce Vars $t_0, t_1, \ldots, t_N$ corresp. to $\tau_1, \ldots, \tau_N$

$$F(t_0, t_1, \cdots) \equiv \langle e^{t_0\tau_0 + t_0\tau_1 + \cdots + t_N\tau_N} \rangle$$

$$= \sum_{n_j \geq 0} \frac{t_0^{n_0} t_1^{n_1} \cdots t_N^{n_N}}{n_0!\, n_1! \cdots n_N!} \langle t_0^{n_0} \tau^{n_1} \cdots \rangle$$

(for example $\langle \tau_3 \tau_4 \tau_3 \rangle = \langle \tau_3^2 \tau_4 \rangle$)

writing the string equ. in this notation:

$$\langle \tau_0 \tau_{d_1} \cdots \tau_{d_n} \rangle = \sum_{i=1}^{n} \langle \tau_{d_1} \cdots \tau_{d_i - 1} \cdots \tau_{d_n} \rangle$$

($\iff$ in terms of F: $\frac{\partial F}{\partial t_0} = \sum_{i \geq 0} t_i \frac{\partial F}{\partial t_i} + \frac{t_0^2}{2}$)

Witten's Conjecture: $F(t_0, t_1, \cdots)$ is uniquely det. by the string equ. and the KdV Hierarchy for $u = \frac{\partial^2 F}{\partial t_0^2}$

$\left(\frac{\partial u}{\partial t_n} = X'_{H_n} = \frac{\partial}{\partial t_0}(d_u H_n) \right.$ and $\left. \frac{d}{dx}(d_u H_n) = \frac{1}{2n+1}\left(u' + 2u\frac{d}{dx} + \frac{\epsilon^2}{4}\frac{\partial^3}{\partial x^3}\right)(d_u H_{n-1}) \right)$

$\Longleftrightarrow$

$$(2n+1)\frac{\partial^3 F}{\partial t_1 \partial t_0^2} = \frac{\partial^2 F}{\partial t_{n-1}\partial t_0}\frac{\partial^3 F}{\partial t_0^3} + 2\frac{\partial^3 F}{\partial t_{n-1}\partial t_0^2} + \frac{\partial^2 F}{\partial t_0} + \frac{1}{4}\frac{\partial^5 F}{\partial t_{n-1}\partial t_0^4}$$

$+$ String eqn.

ex) restrict eqn. to $t=0$:

$$\left.\frac{\partial F}{\partial t_0^{n_0}\partial t_1^{n_1}\cdots\partial t_N^{n_N}}\right|_{t=0} = \langle \tau_0^{n_0}\cdots\tau_N^{n_N}\rangle$$

we get $(2n+1)\langle\tau_n\tau_0^2\rangle\cancelto{=1}{\langle\tau_0^3\rangle} + \cancelto{0}{2\langle\tau_{n-1}\tau_0^2\rangle}$

$+\frac{1}{4}\langle\tau_{n-1}\tau_0^4\rangle$

$3g-3+3=n \Rightarrow n=3g$

$(6g+1)\cdot\langle\tau_{3g}\tau_0^2\rangle_g = \langle\tau_{3g-1}\tau_0\rangle_g + \frac{1}{4}\langle\tau_{3g-1}\tau_0^4\rangle_{g-1}$

$=(\text{string eqn}) = \langle\tau_{3g-2}\rangle_g\cdot(6g+1) = \langle\tau_{3g-2}\rangle_g + \frac{1}{4}\langle\tau_{3g-5}\rangle_{g-1}$

$= \cdots$ recursion reln.

$$\boxed{=\frac{1}{24^g g!}}$$