Loop spc. formalism 1 | Givental 7/22/8

Genus-0 Theory:

$$\mathcal{F}_0(t) = \sum_{n,d} \frac{Q^d}{n!} \langle t(\psi), \dots, t(\psi) \rangle_{0,n,d}$$

$$F(\tau) = \sum_{n,d} \frac{Q^d}{n!} \langle \tau, \dots, \tau \rangle_{0,n,d}$$

$$J_\alpha = \langle\langle \frac{\phi_\alpha}{z-\psi} \rangle\rangle = \sum \frac{Q^d}{n!} \langle \frac{\phi_\alpha}{z-\psi}, \tau, \dots, \tau \rangle_{0,n+1,d} + (\phi_\alpha, \tau) + (\phi_\alpha, 1) z$$

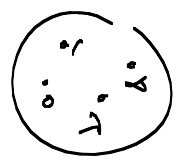
$$S^\alpha_\beta = \langle\langle \phi^\alpha, \frac{\phi_\beta}{z-\psi} \rangle\rangle$$

$$V^\alpha_\beta = \langle\langle \frac{\phi^\alpha}{x-\psi}, \frac{\phi_\beta}{z-\psi} \rangle\rangle$$

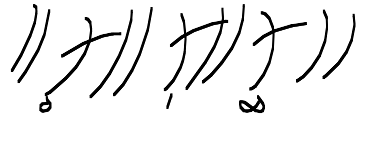
str equs:

$$\left\{ \begin{aligned} \partial_1 F &= \frac{(\tau,\tau)}{2} \\ \partial_1 J &= \frac{1}{z} J \\ \partial_1 S &= \frac{1}{z} S \\ \partial_1 V &= \left(\frac{1}{x}+\frac{1}{y}\right) V \end{aligned} \right\}$$

$$\begin{array}{c} X_{g,n,d} \\ \downarrow ct \\ \overline{\mathcal{M}}_{g,n} \end{array}$$

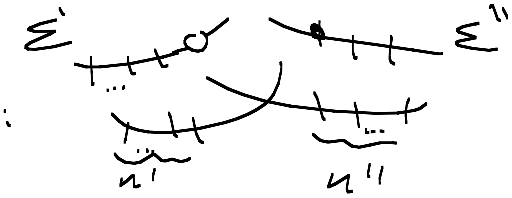
ex)  $\mathbb{P}^1$ $\overline{M}_{0,4}$

$0, 1, \infty$ corresp. to curves w/ nodes

in $\overline{M}_{0,4}$  $\dots = \overline{M}_{0,4}$

WDVV-eqns:

Look at inverse image of $\circ$:



$$\coprod_{\substack{d'+d''=d\\ n'+n''=n}} \left(X_{0,3+n',d'} \times X_{0,3+n'',d''} \right) \cap \Delta \quad \text{gives } \phi_{\alpha}, \phi_{\beta}$$

$$\begin{array}{ccc} & \left(X_{0,3+n',d'} \times X_{0,3+n'',d''} \right) & \\ & \downarrow{\scriptstyle ev_0 \times ev_\bullet} & \\ & X \times X & \end{array}$$

want to study:

$$\sum_{n',n''} \sum_{d',d''} \sum_{\mu} \langle \phi_\alpha, \phi_\beta, \underbrace{\tau, \ldots, \tau}_{n'}, \phi_\mu \rangle_{0,3+n',d'} \cdot \binom{n}{n'} \frac{Q^{d'+d''}}{n!} \cdot \langle \phi^\mu, \underbrace{\tau, \ldots, \tau}_{n''}, \phi_\alpha, \phi_\beta \rangle_{0,3+n'',d''}$$

Denote

$$\partial_{\alpha\beta\gamma}(F) = F_{\alpha\beta\gamma}$$

Then

WDVV

$$\boxed{F_{\alpha\beta\mu} F^{\mu}_{\gamma\delta}(\tau) \text{ is totally symmetric w.r.t. } \alpha\beta\gamma\delta}$$

so,

$$\sum_{\mu} F^{\alpha}_{\beta\mu} F^{\mu}_{\gamma\delta} = \sum F^{\alpha}_{\gamma\mu} F^{\mu}_{\beta\delta}$$

$$(\phi_\beta \bullet)(\phi_\gamma \bullet) = (\phi_\gamma \bullet)(\phi_\beta \bullet) \qquad (\phi_\alpha \bullet)\phi_\beta =: \phi_\alpha \bullet \phi_\beta$$

$$\phi_\beta \bullet (\phi_\gamma \bullet \phi_\delta) = \phi_\gamma \bullet (\phi_\beta \bullet \phi_\delta)$$

gives an assoc., commutative mult of $H = H^*(X, \mathbb{Q})$

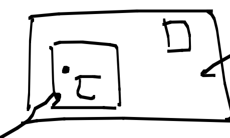
which depends on $\tau \in H$.

$\nearrow$ w/ $\mathbb{Q} = \mathbb{Q}[[Q]]$

<u>note</u> $(\phi_\alpha \bullet \phi_\beta, \phi_\gamma) = F_{\alpha\beta\gamma}$

$= (\phi_\alpha, \phi_\beta \bullet \phi_\gamma)$

"this is" Big Quantum Cohomology.



linear spc. considered as a manifold.

tangent spc. at τ has assoc. mult. $\bullet$

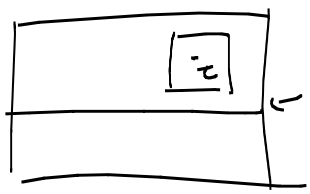
recall divisor eqn:

$$\frac{\partial}{\partial \tau^i}(\cdots) = Q_i \frac{\partial}{\partial Q_i}(\cdots)$$

$\{P_i\} \subset H^2(X)$ basis.

$Q_i e^{\tau^i}$

(geometric interpretation)



H

$\leftarrow H^2(X)$

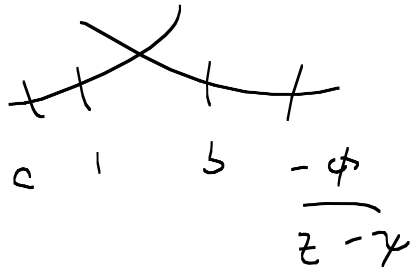
(so vectors in H are denoted by their coordinates $\frac{\partial}{\partial \tau^i}$

Consider connection on this space $= T(\text{space})$

(metric is given by Poincaré Pairing)

$$\nabla_z \equiv d + \frac{1}{z}\sum\left(\frac{\partial}{\partial \tau^\alpha}\bullet\right) d\tau^\alpha \wedge$$

$\nabla_z^2 = 0$ $\quad \phi^\alpha \quad \frac{\partial}{\partial \tau^\alpha}$



$$F_{ab}^{\mu}(\tau) \ll \phi_\mu, 1, \frac{\phi}{z-\chi} \gg (\tau)$$

$$= F_{ab}^{\mu} \ll \phi, b, \frac{\phi}{z-\chi} \gg$$

get $\underset{(b\bullet)}{\underset{\shortparallel}{(\partial_b \bullet)}} \cancel{\tfrac{1}{2}}S = \partial_b S$

$\Rightarrow$ O fundamental soln. to $\nabla S = 0$.

: skip

$\mathcal{F}_0(t_0, t_1, \ldots,)$ $\quad t_k = \Sigma t_k^\alpha \phi_\alpha$

$\{\phi_\alpha\}$ $\boxed{1 \in H, (,)}$

$Q = \mathbb{Q}[[Q]]$

$f, g \in \mathcal{H} = H((\frac{1}{z}))$ Laurent series

symplectic loop spc.

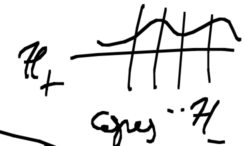
$$\Omega = \frac{1}{2\pi i} \oint (f(-z), g(z))\, dz$$

$$f = \cdots \frac{-p_\alpha^2 \phi^\alpha}{z^3} + \frac{p_\alpha^1 \phi^\alpha}{z^2} - \frac{p_\alpha^1 \phi^\alpha}{z} + \varepsilon_0^\alpha \phi_\alpha + \varepsilon_1^\alpha \phi_\alpha z + \varepsilon_2^\alpha \phi_\alpha z^2$$

$+ - -$

$$\Omega\left(\frac{\phi^\alpha}{(-z)^{k+1}}, \phi_\alpha z^k\right)\left(\frac{\phi_\alpha}{z^{k+1}}, \phi_\alpha z^k\right) = \frac{1}{z}$$

graph of $\mathcal{F}_0$ realized in $\mathcal{H}$:



$t(z) = t_0 + t_1 z + t_2 z^2 + \dots$

$\mathcal{F}_0 : \mathcal{H}_+ \to \mathbb{Q} \qquad = H[z].$

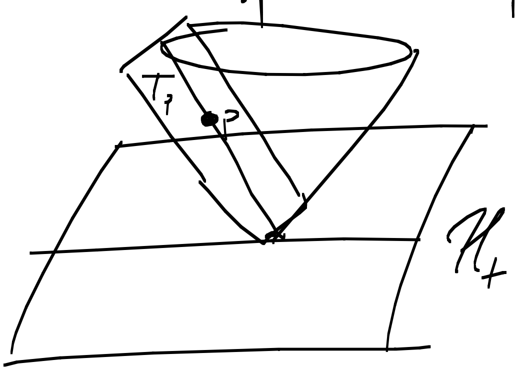
$$\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_- = T^*\mathcal{H}_+$$

$\mathcal{H}_+$: Lagrangian subspace

$\mathcal{H}_-$: transversal Lagrangian subspace.

$(\mathcal{H}, \Omega) = T^*\mathcal{H}_+ \supset \mathcal{L} = \{(p,q)\}$

$p = d_{1\cdot z+q}\mathcal{F}_0(z = 1\cdot z + q)$



$\mathcal{L}$ is an <u>overruled</u> <u>Lagrangian cone</u>

1) <u>overruled</u>: 1) T is tangent to $\mathcal{L}$ exactly along zT : which means:

1) $zT \subset T$

2) $zT \in \mathcal{L}$

3) Tan. sp. along zT are $z^2T \subset zT \subset \frac{1}{z}T \subset \frac{1}{z^2}T \dots = T.$

Dilaton equ: $\mathcal{L}_E \mathcal{F}_0 = 2\mathcal{F}_0$

String Eqn $\left[f \longmapsto \frac{1}{z} f\right]$ is anti-symmetric wrt. Ω.

$$\Omega(zf, g) + \Omega(f, zg) = 0$$

$\Omega(f, \frac{1}{z} f)$

- In GW theory, $\mathcal{L}$ is an over-ruled Lagrangian Cone.

$$L^{(z)}GL(H) = \{ M(z) \mid M^*(-z)\, M(z) = 1 \}$$

twisted Loop group.

$$\Omega(Mf, Mg) = \Omega(f, g)$$