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Most basic defn. of a TFT:

(in dim n) is a functor from $Cob_n^B \xrightarrow{F} \mathcal{C}$ such that

$\mathcal{C}$ is a monoidal category, F is a tensor functor

$Cob_n^B :=$ category $Ob = \{$n-1 mnflds which are closed, cpct. w/ B-structure$\}$

morphisms: $Hom_{Cob_n^B}(M^{n-1}, N^{n-1}) = \left\{ \begin{array}{c} n \text{ mnflds w/} \\ \text{struct } B \\ \text{and } \partial W \cong W \amalg \bar{N} \end{array} \right\} /_{\text{diffeo}}$

assume $\bar{N}$ means something wrt. B.

(2) Defn A tensor cat. $\mathcal{C}$ has the structure of a tensor product functor

$$\mathcal{C} \times \mathcal{C} \xrightarrow{\otimes} \mathcal{C}$$

(for now assume always symmetric + assoc.)

ex) Sets w/ $\otimes = \amalg, \times$ possible tensor structures

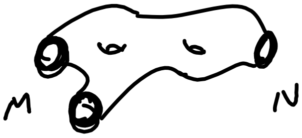
ex) Cob_n^B w/ $\amalg$

ex) $Vect_k$ w/ $\otimes = \otimes_k$

⋮

First example of a TFT: $F: Cob_n^B \longrightarrow (\text{chain cmplxes}/_R, \overset{\amalg}{\otimes_R})$

first attempt: try $F \overset{?}{=} C^*(\bullet)$ singular cochains



$C^*(M) \dashrightarrow C^*(N)$

have:
$$\begin{tikzcd} & W & \\ M \arrow[ur] & & N \arrow[ul] \end{tikzcd}$$
get
$$\begin{tikzcd} & C^*W \arrow[dl] \arrow[dr] & \\ C^*M \arrow[ur, dashed, bend left, "v_*"'] & & C^*N \end{tikzcd}$$

want $j_*: C^*M \to C^*W$.

<u>Idea</u>: want pull-push over a cobordism.
look for situations where we can do push-pull.

<u>Idea'</u> increase the categorical level. (for example at some level)

Let $\mathcal{C}$ be a <u>k-linear category</u>. categories enriched in chain complexes over R.

and $\exists \otimes$ for these.

<u>2nd attempt.</u> $F = C^*(\bullet)\text{-mod}. \; : X \mapsto C^*(X)\text{-mod}.$

$$= \left\{ \begin{array}{l} \text{cat. w/} \\ \text{Obs} = M \text{ complex}_R \text{ w/} \\ \quad A \underset{R}{\otimes} M \to M \text{ an assoc. action and maps of complexes} \end{array} \right\}$$

try again:

$$M^{n-1} \longrightarrow W^n \longleftarrow N^{n-1}$$

now we can push-pull:

suppose $A \xrightarrow[f]{} B$ map of DGAs,

$$\begin{tikzcd} A\text{-mod} \arrow[r, shift left, "\mathrm{Ind}_f"] & B\text{-mod} \arrow[l, shift left, "\mathrm{Res}_f"] \end{tikzcd}$$

$\mathrm{Ind}_f = B \underset{A}{\otimes} (-)$

$\mathrm{Res}_f =$ obvious.

$$\begin{tikzcd} & C^*(W)\text{-mod} \arrow[dr, "\mathrm{Ind}"] & \\ C^*(M)\text{-mod} \arrow[ur, "\mathrm{Res}"] \arrow[rr, dashed, bend right, "\mathrm{Ind} \circ \mathrm{Res}"'] & & C^*(N)\text{-mod}. \end{tikzcd}$$

✓

But note that we really only started w/ $[W]$ a diffeo class.

<u>Second definition</u>: let $\overline{\mathrm{Cob}}_n = \left\{ \begin{array}{l} \text{ob} := \{(n-1)\text{-mnflds}\} \\ \text{mor} := \mathrm{Hom}(M,N) = \text{classifying spc. for bordisms btwn } M,N \text{ w/ extra struct} \end{array} \right.$

a topological cat

Allows us to see diffeomorphism grp. actions.

another formulation: needed properties of $\mathrm{Hom}(M,N)$: (is a space)

$$\pi_0 \underset{\mathrm{Cob}_n^B}{\mathrm{Hom}}(M,N) = \mathrm{Hom}_{\mathrm{Cob}_n^B}(M,N)$$

$$\mathrm{Hom}(M,N) \simeq \coprod_{W,\ [W]\in \mathrm{Hom}_{\mathrm{Cob}}(M,N)} B\mathrm{Diff}(W)$$

Then, $\mathrm{Map}(X, \mathrm{Hom}(M,N))$, $X \xrightarrow{f} \coprod_{[W]} B\mathrm{Diff}(W)$. If X is connected, it hits only one component, say $[W]$. $X \xrightarrow{f} B\mathrm{Diff}(W)$.

given a map $f: X \to B\mathrm{Diff}(W)$,

$$\begin{tikzcd} f^*E\mathrm{Diff} \arrow[r] \arrow[d] \arrow[loop above, "\mathrm{Diff}W"] & E\mathrm{Diff}(W) \arrow[d] \arrow[loop above, "\mathrm{Diff}(W)"] & \mathrm{Diff}(W) \arrow[l] \arrow[d] \\ X \arrow[r, "f"] & B\mathrm{Diff}(W) & \{*\} \arrow[l, "\forall pt"] \end{tikzcd}$$

Take the associated bundle w/ fibers W and structure grp $\mathrm{Diff}(W)$

so, get this universal bundle over $B\mathrm{Diff}(W)$:

$$\begin{tikzcd} E \arrow[d] \arrow[loop above, "\mathrm{Diff}W"] & W \arrow[l] \arrow[d] \\ B\mathrm{Diff}(W) & * \arrow[l] \end{tikzcd}$$

pullback E:

$$\begin{tikzcd} W \arrow[r] \arrow[d] & f^*E \arrow[r] \arrow[d] \arrow[loop above, "\mathrm{Diff}W"] & E \arrow[d] \\ * \arrow[r] & X \arrow[r, "f"] & B\mathrm{Diff}(W) \end{tikzcd}$$

f^*E is a smooth family of bordisms equiv. to W.

So, maps $X \to B\mathrm{Diff}(W)$ classify bordisms.

Maps $(X, \mathrm{Ham}(M,N))$ gives X-families of bordisms btwn M, N (w/ smooth transition functions)

previous example: $F = C^*(\cdot)$-mod,

on $F(M) = C^*(M)$-mods $\circlearrowright$ $\mathrm{Diff}(M)$.

$\circlearrowright$ $\mathrm{Aut}^{ho}(M)$ which are homologic to a homotopy equivalence

$\cap$

$\mathrm{Maps}(M,M)$

Has lots of symmetries, but not of the most refined kind.

Choose some bordism W, btwn M, N, get a map

$\{\text{Space of bordisms}\} \to \{\text{Functors from } C^*(M)\text{-mod} \to C^*(N)\text{-mod}\}$

i.e. there is a map from $B\mathrm{Diff}(W) \longrightarrow \underbrace{\mathrm{Fun}(C^*(M)\text{-mod}, C^*(N)\text{-mod})}_{\text{has the structure of a space.}}$

most basic example

[If $\mathrm{Fun}(D,D')$ is a category, discard non-invertible nat'l transformations
D, D' ord. cats

$\Rightarrow$ get groupoid

take the classifying space $B : \mathcal{G} \to B\mathcal{G}$

(explicitly, $|N(\mathcal{G})|$)

have $\mathrm{Cob}_n \longrightarrow \{k\text{-linear categories}, \oplus\}$

tensor functor? : $F(M \amalg N) \simeq C^*(M \amalg N)\text{-mod} \overset{?}{\simeq} C^*(M) \otimes C^*(N)\text{-mod}$

cheat: didn't precisely define $\mathrm{Hom}(M,N)$

$$\begin{tikzcd}[column sep=small]
 & W & & W' & & W'' & \\
M \arrow[ur] & & N \arrow[ul] \arrow[ur] & & O \arrow[ul] \arrow[ur] & & P \arrow[ul]
\end{tikzcd}$$

$B\mathrm{Diff}(W) \times B\mathrm{Diff}(W') \times B\mathrm{Diff}(W'') \to \mathrm{Hom}(M,P)$

to give diffeomorphism, need, e.g. a trivialization of the diff'l in a nbhd of the ∂dy.

geometric analog of the previous example:

$\mathcal{C}$ is k-linear category. We used before that we could "do something" to $\mathcal{C}$ given M. Another thing to do is $\otimes \mathcal{C}$ w/ a space:

$M \otimes \mathcal{C} :=$ (general defn: let $\mathcal{X}$ be a top. cat. M a space. $\forall X, Y \in \mathcal{X}$, $\underline{\mathrm{Hom}}_{\mathcal{X}}(X,Y)$ is a space,

then $M \otimes X$ is an object in $\mathcal{X}$ w/ an equiv:

$\underline{\mathrm{Hom}}(M \otimes X, Y) \xrightarrow{\sim} \mathrm{Map}(M, \underline{\mathrm{Hom}}(X,Y))$) $\forall Y$

ex) $\mathcal{C}$ = spaces $\quad M \otimes X = M \times X$

ex) $\mathcal{C}$ = sets $\quad M \otimes X = \pi_0(M) \times X$

ex) $\mathcal{C}$ = chain complexes "ch", $\mathrm{Hom}_{ch}(X,Y)$ has ch. cplx struct

$\exists$ an adjunction: $\quad \mathrm{Ch} \underset{\text{DOLD KAN}}{\overset{\Gamma}{\rightleftarrows}} s\mathrm{Ab} \underset{\text{free ab gp}}{\overset{\text{forget}}{\rightleftarrows}} s\mathrm{Set} \underset{\mathrm{Sing.}}{\overset{|\cdot|}{\rightleftarrows}} \mathrm{Top}.$

So, Spc. $\mathrm{Hom}(X,Y) = |\Gamma \, \mathrm{Hom}_{ch}(X,Y)|$

note: $|\Gamma(-)|$ is adjoint to C_*.

aside: (def. $Q = |\Gamma(-)|$) if X is ch. cmplx

$$\pi_n Q(X) = \begin{cases} H_n X & n \geq 0 \\ 0 & n < 0 \end{cases}$$

then $M \in$ Space, X ch. cplx $M \otimes X = C_*(M) \underset{k}{\otimes} X$

thm set $\mathcal{X} \underset{\text{def}}{=} \{k\text{-linear categories}\}$ forms a topological category

define $M \otimes \mathcal{C}$ for $\mathcal{C} \in \text{ob}\,\mathcal{X}$ as on the left board.

approaches a field theory:

$$\left(\begin{tikzcd}[column sep=small] & W & \\ M \arrow[ur] & & N \arrow[ul] \end{tikzcd} \right) \otimes \mathcal{C} \rightsquigarrow \begin{tikzcd} & W \otimes \mathcal{C} \arrow[dl, dashed, shift left] \arrow[dr, dashed, shift left, "j^!"] & \\ M \otimes \mathcal{C} \arrow[ur, "i"] \arrow[rr, dashed] & & N \otimes \mathcal{C} \arrow[ul] \end{tikzcd}$$

$j^! \circ i$

tensors have adjoints

ex) $\mathcal{X} =$ cat of $\left\{ \begin{array}{l} k\text{-linear tensor categories} \\ \text{w/ } \otimes\text{-functors} \end{array} \right\}$

note a k-linear cat is enriched and tensored over chain complexes over k. as before

and presentable.

in particular, have small limits + colimits (homotopy)

$\mathcal{C} = A\text{-mod}$, w/ A commutative DGA

$\otimes = \bullet \overset{\mathbb{L}}{\underset{A}{\otimes}} \bullet$

subexample: $A = C^*(K)$ cochains on a space.

For $\mathcal{C} = C^*(K)\text{-mod}$ $M \otimes \mathcal{C} = C^*(\text{Map}(M, K))\text{-mod}.$

$\left\{ \begin{array}{c} k\text{-linear} \\ \otimes\text{-cat} \end{array} \right\} \xleftarrow{\text{mod}}$ Comm DGA

is a top. cat.