

John Francis 5

$M$  spc.,  $C$  symmetric nonunital  $k$ -linear cat.

describe  $M \otimes C$ :

$\{ \text{Sym nonunital } k\text{-cats} \}$

$\uparrow$  mod

Comm DGA's  $\longleftarrow$  Spaces

$C^*(-, k)$

Caveat: either

a) check  $k=0$

b) replace comm. w/  $E_\infty$

Describe  $M \otimes A$ : for  $A \in \text{Comm DGA}$  ex:

$\bullet A = \text{Sym}^* V \Rightarrow M \otimes A = \text{Sym}^*(M \otimes V) = \text{Sym}^*(C_* M \otimes V)$

$\bullet H = C^*(X) \Rightarrow$

$M \otimes A \cong C^*(\text{Map}(M, X))$

note: 
$$\begin{array}{ccc} S^1 & \xrightarrow{*} & * \\ \downarrow & \text{hoP} & \downarrow \\ * & \xrightarrow{} & S^1 \end{array}$$
 homotopy presert

(generally, 
$$\begin{array}{ccc} X & \xrightarrow{*} & * \\ \downarrow \text{Ho} & & \downarrow \\ * & \xrightarrow{} & \Sigma X \end{array}$$
)

first let  $M =$  discrete spc. or a finite set even.

if  $M = *$  then  $* \otimes A \cong A$

since  $\otimes$  has the universal prop  $\text{Hom}(* \otimes A, B) \cong \text{Hom}_{\text{spc}}(*, \underline{\text{Hom}}(A, B)) = \underline{\text{Hom}}(A, B)$

similarly, for  $M = I$  set,

$I \otimes A \cong A^{\otimes |I|}$

observe: for  $A \in \text{Comm DGA}$   
spaces  $\longrightarrow$  Comm DGA  
 $(-) \otimes A$

: this functor preserves colimits.

If  $\{K_\alpha\}$  a diagram of spaces,

$\text{Hom}(\text{Colim}_\alpha K_\alpha, \underline{\text{Hom}}(A, B)) \longleftarrow$  preserves colimits in sources

$\downarrow$   
 $\text{Lim}_\alpha (K_\alpha, \underline{\text{Hom}}(A, B)) \Rightarrow (-) \otimes A$  must also.

This fact is useful: ex)  $M = S'$   $S' \otimes A = ?$  resolved on  $D'$

$$S' = \text{Colim} \left( \begin{array}{ccc} S^0 & \rightarrow & D' \\ \downarrow & & \\ & & D' \end{array} \right) \otimes A \cong \text{Colim} \left( \begin{array}{ccc} S^0 \otimes A & \rightarrow & A \\ \downarrow & & \\ & & A \end{array} \right) \text{equiv. to } A$$

$$= \text{Colim} \left( \begin{array}{ccc} A \otimes A & \rightarrow & A \\ \downarrow & & \\ & & A \end{array} \right)$$

Portents in  $\text{ComDGA}$  are computed as tensor products.

$$= \underbrace{A \otimes A}_{A \otimes A} =: \underline{\underline{\text{TH}(A)}}$$

set  $A = C^*(X)$  want to understand  $M \otimes A$

need non-trivial fact: spaces  $C^*(-) \rightarrow \text{ComDGA}$

sends homotopy pullbacks to homotopy pullbacks

ex):  $\begin{array}{ccc} \Omega X \rightarrow * & \text{homotopy pullback} & \\ \downarrow \text{hoPull} \downarrow & & \\ * \rightarrow X & \cong & \Omega X \rightarrow F(*) \end{array}$   $\begin{array}{l} \text{Fibrant replacement} \\ = PX \\ \downarrow \text{Pull} \downarrow \\ * \rightarrow X = \Omega X \rightarrow PX \\ \downarrow \text{Pull} \downarrow \\ * \rightarrow X \end{array}$

$$\text{so, } C^* \left( \begin{array}{ccc} P \rightarrow Y \\ \downarrow \text{ho} \downarrow \\ X \rightarrow Z \end{array} \right) \Rightarrow C^*(P) \cong C^*(X) \otimes_{C^*(Z)} C^*(Y)$$

consequence:  $M \otimes C^*(X) \cong C^*(\text{Maps}(M, X))$

why: step 1: True for  $M$  discrete spc.  $|I| < \infty$ .

ie: LHS =  $C^*(X)^{\otimes |I|}$ , RHS =  $C^*(X)^{\otimes |I|} \cong C^*(X)^{\otimes |I|}$   $\text{K\"onnethe Familien}$

step 2: Both sides preserve homotopy colimits

LHS already checked  $\checkmark$

RHS let  $M = \text{Colim}(M_\alpha)$

$$C^*(\text{Map}(\text{colim } M_\alpha, X)) = C^*(\text{lim}_\alpha (\text{Map}(M_\alpha, X))) = \text{colim}_\alpha C^*(\text{Map}(M_\alpha, X))$$

step 3: All spaces are quasi'd up to homotopy by homotopy colimits of finite sets.

(CW approximation)  $\blacksquare$

ex)  $HM_+(C^+(X)) \cong H^1(\text{Map}(S^1, X))$  ( $X$  simply connected)

{ sym. nonoidal  $k$ -cats }

↑ mod-  
(con)DA

claim:  $M \otimes A\text{-mod} \cong (M \otimes A)\text{-mod}$ .

pf  $\text{Fun}^{\otimes}(A\text{-mod}, \mathcal{C}) \cong \text{Hom}(A, \text{End}_{\mathcal{C}}(1))$   
 any tensor functor (and colim preserving)  
 sends  $A \mapsto 1_{\mathcal{C}}$  so get  
 $\text{Hom}(A, A) \rightarrow \text{End}(1_{\mathcal{C}})$   
 $\cong \text{Hom}_{\text{con}DA}(A, A)$

Then, since our functor is colimit-preserving and every elt. in  $A\text{-mod}$  is gen'd by colimits involving  $A \Rightarrow$  functor determined by its value on  $A$ .

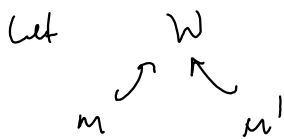
check:  $\text{Hom}(M \otimes A\text{-mod}, \mathcal{C}) \cong \text{Hom}(M, \text{Hom}(A\text{-mod}, \mathcal{C})) \cong$   
 $\cong \text{Hom}(M, \text{Hom}_{\text{con}DA}(A, \text{End}_{\mathcal{C}}(1))) \cong \text{Hom}_{\text{con}DA}(M \otimes A, \text{End}_{\mathcal{C}}(1))$   
 $\cong \text{Hom}((M \otimes A)\text{-mod}, \mathcal{C})$  ■

General conclusion  $M \otimes C^+(X)\text{-mod} \cong C^+(\text{Map}(M, X))\text{-mod}$ .

Return to TFTS: let  $\mathcal{C}$  be a symmetric tensor cat. over  $k$   
 let's define a fld. thry:

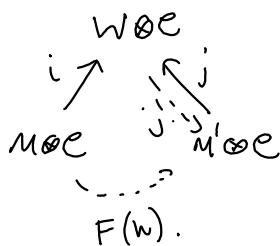
want:  $\widetilde{\text{Cob}}_n \longrightarrow \{ k\text{-linear cats.} \} \longleftarrow (\omega, 1)\text{-cat.}$

$M^{n-1} \longmapsto M \otimes \mathcal{C}$



be a coherdism, want  $F(w): M \otimes \mathcal{C} \rightarrow M' \otimes \mathcal{C}$

immediately get



now  $j$  is a colim-preserving functor, so, by general stuff, it has a right adjoint,  $j!$ .

Set  $F(w) := i \circ j!$

$F(w)$  is not a tensor functor.

ex)  $X$  is a  $\mathbb{C}$ -c.

$$X \xrightarrow{f} *$$

$\Gamma$  adjoint of  $f$   
 $\Gamma(k \otimes k') \neq \Gamma(k) \otimes \Gamma(k')$

$\Gamma$  doesn't preserve  $\otimes$  :  $\Gamma(k \otimes k') \neq \Gamma(k) \otimes \Gamma(k')$

check: is  $F$  a tensor functor?

$$F(M \amalg N) = (M \amalg N) \otimes e = (M \otimes e) \otimes (N \otimes e) \checkmark$$

$$\begin{aligned} \text{ex) : } C^*(\text{Map}(M \amalg N, X))\text{-mod} &= C^*(\text{Map}(M, X) \times \text{Map}(N, X))\text{-mod} \\ &\stackrel{\text{K\"unneth}}{\cong} [C^*(\text{Map}(M, X)) \otimes C^*(\text{Map}(N, X))]\text{-mod} \\ &\cong C^*(\text{Map}(M, X))\text{-mod} \otimes C^*(\text{Map}(N, X))\text{-mod} \end{aligned}$$

(this is the basic fact about  $\otimes$  of  $k$ -lin (cts):  
 $A \otimes B\text{-mod} \cong A\text{-mod} \otimes B\text{-mod}$ )

$\Rightarrow$  defines a field theory.

Q: what was wrong w/ the  $F : M \mapsto C^*(M)\text{-mod}$   
 $F(M \amalg M') \neq F(M) \otimes F(M')$



