John Francis $\boxed{5}$ $\quad M$ spc., $\mathcal{C}$ symmetric monoidal k-linear cat.

describe $M \otimes \mathcal{C}$:

$$\begin{array}{ccc} & \{\text{Sym monoidal } k\text{-cats}\} & \\ & \uparrow \text{mod} & \\ & \text{Comm DGAs} & \xleftarrow{C^*(-,k)} \text{Spaces}\end{array}$$

caveat: either
a) char $k = 0$
b) replace comm. w/ E_∞

Describe $M \otimes A$: for $A \in$ Comm DGA ex!

- $A = \mathrm{Sym}^* V \Rightarrow M \otimes A = \mathrm{Sym}^*(M \otimes V) = \mathrm{Sym}^*(C_* M \otimes_k V)$

- $A = C^*(X) \Rightarrow$

$M \otimes A \simeq C^*(\mathrm{Map}(M, X))$,

note:

$$\begin{tikzcd} S^1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{hoP}"] & * \arrow[d] \\ * \arrow[r] & S^1 \end{tikzcd}$$

homotopy pushout

(generally,
$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{Ho}"] & * \arrow[d] \\ * \arrow[r] & \Sigma X \end{tikzcd}$$
)

first let M = discrete spc. or a finite set even.

If $M = *$ then $* \otimes A \simeq A$ since $\otimes$ has the universal prop
$\mathrm{Hom}(* \otimes A, B) \simeq \mathrm{Hom}_{spc}(*, \underline{\mathrm{Hom}}(A,B))$
$= \underline{\mathrm{Hom}}(A,B)$

similarly, for $M = I$ set,

$I \otimes A \simeq A^{\otimes |I|}$

observe: for $A \in$ Comm DGA

$\text{spaces} \xrightarrow{(-) \otimes A} \text{Comm DGA}$: this functor preserves colimits.

If $\{K_\alpha\}$ a diagram of spaces,

$\mathrm{Hom}(\mathrm{colim}_\alpha K_\alpha, \underline{\mathrm{Hom}}(A,B))$ ← preserves colimits in source

$\shortparallel$

$\lim_\alpha (K_\alpha, \underline{\mathrm{Hom}}(A,B)) \Rightarrow (-) \otimes A$ must also.

This fact is useful: ex) $M = S^1$ $\qquad S^1 \otimes A = ?$ — resolved on D^1 equiv. to A

$$S^1 = \operatorname{Colim}\left(\begin{tikzcd} S^0 \arrow[r] \arrow[d] & D^1 \\ D^1 & \end{tikzcd}\right) \otimes A \simeq \operatorname{Colim}\left(\begin{tikzcd} S^0\otimes A \arrow[r] \arrow[d] & A \\ A & \end{tikzcd}\right)$$

$$= \operatorname{Colim}\left(\begin{tikzcd} A\otimes A \arrow[r] \arrow[d] & A \\ A & \end{tikzcd}\right)$$

$\Longleftarrow$ Pushouts in ComDGA are computed as tensor products.

$$= A \overset{\mathbb{L}}{\underset{A\otimes A}{\otimes}} A =: \underline{\underline{HH_*(A)}}$$

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set $\quad A = C^*(X) \quad$ want to understand $M \otimes A$

need nontrivial fact: $\quad \text{spaces} \xrightarrow{C^*(-)} \text{ComDGA}$

sends homotopy pullbacks to homotopy pushouts

ex):

$$\begin{tikzcd} \Omega X \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{hoPull}"] & * \arrow[d] \\ * \arrow[r] & X \end{tikzcd} \quad \text{homotopy pullback.} \quad \simeq \quad \begin{tikzcd} \Omega X \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{Pull}"] & F(*) \arrow[d] \\ * \arrow[r] & X \end{tikzcd} = \begin{tikzcd} \Omega X \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{Pull}"] & PX \arrow[d] \\ * \arrow[r] & X \end{tikzcd}$$

Fibrant replacement $= PX$

$$\text{so,} \quad C^*\left(\begin{tikzcd} P \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{Ho}"] & Y \arrow[d] \\ X \arrow[r] & Z \end{tikzcd}\right) \Longrightarrow C^*(P) \simeq C^*(X) \overset{\mathbb{L}}{\underset{C^*(Z)}{\otimes}} C^*(Y)$$

consequence: $\quad M \otimes C^*(X) \simeq C^*(\mathrm{Maps}(M, X))$

why: step 1: True for M discrete spc. $|I| < \infty$.

i.e.: LHS $= C^*(X)^{\otimes |I|}$, RHS $= C^*(X^{|I|}) \simeq C^*(X)^{\otimes |I|}$ Künneth Formula

step 2: Both sides preserve homotopy colimits

LHS already checked ✓

RHS let $M = \operatorname{Colim}(M_\alpha)$

$$C^*(\mathrm{Map}(\operatorname{colim} M_\alpha, X)) = C^*\left(\lim_\alpha (\mathrm{Map}(M_\alpha, X))\right) = \operatorname{colim}_\alpha C^*(\mathrm{Map}(M_\alpha, X))$$

step 3: All spaces are generated up to homotopy by homotopy colimits of finite sets.

(CW approximation) ∎

ex) $HH_*(C^*(X)) \cong H^*(\mathrm{Map}(S^1, X))$ (X simply connected)

$$\begin{tikzcd}
\{\text{sym. monoidal } k\text{-cats}\} \\
\mathrm{CommDGA} \arrow[u, "\mathrm{Mod}-"']
\end{tikzcd}$$

claim: $M \otimes A\text{-mod} \simeq (M \otimes A)\text{-mod}$.

pf $\underbrace{\mathrm{Fun}^{\otimes}}_{\substack{\text{any tensor functor (and colim preserving)}\\ \text{sends } A \mapsto 1_{\mathcal{C}} \text{ so get}}}(A\text{-mod}, \mathcal{C}) \cong \mathrm{Hom}_{\mathrm{CommDGA}}(A, \mathrm{End}_{\mathcal{C}}(1))$

$$\underset{\substack{\| \\ A}}{\mathrm{Hom}_{A\text{-mod}}(A,A)} \longrightarrow \underset{\text{map of CommDGA}}{\mathrm{End}(1_{\mathcal{C}})}$$

Then, since our functor is colimit-preserving and every elt. in A-mod is gen't'd by colimits involving A $\Rightarrow$ functor determined by its value on A.

check: $\underline{\mathrm{Hom}}(M \otimes (A\text{-mod}), \mathcal{C}) \cong \mathrm{Hom}(M, \underline{\mathrm{Hom}}(A\text{-mod}, \mathcal{C})) \cong$

$\cong \mathrm{Hom}(M, \underline{\mathrm{Hom}}_{\mathrm{CommDGA}}(A, \mathrm{End}_{\mathcal{C}}(1))) \cong \underline{\mathrm{Hom}}_{\mathrm{CommDGA}}(M \otimes A, \mathrm{End}_{\mathcal{C}}(1))$

$\cong \underline{\mathrm{Hom}}((M \otimes A)\text{-mod}, \mathcal{C})$ $\blacksquare$

General conclusion $\underline{M \otimes C^*(X)\text{-mod}} \simeq \underline{C^*(\mathrm{Map}(M, X))\text{-mod}}$.

Return to TFTS: Let $\mathcal{C}$ be a symmetric tensor cat. over k
let's define a fld. thry:

want: $\widetilde{\mathrm{Cob}}_n \longrightarrow \{k\text{-linear Cat.}\} \longleftarrow ((\infty,1)\text{-cat.})$

$M^{n-1} \longmapsto M \otimes \mathcal{C}$

Let
$$\begin{tikzcd}
 & W & \\
M \arrow[ur] & & M' \arrow[ul]
\end{tikzcd}$$
be a cobordism, want $F(W) : M \otimes \mathcal{C} \to M' \otimes \mathcal{C}$

immediately get
$$\begin{tikzcd}
 & W \otimes \mathcal{C} \arrow[dr, dashed, bend left, "j^!"] & \\
M \otimes \mathcal{C} \arrow[ur, "i"] \arrow[rr, dashed, bend right, "F(W)."'] & & M' \otimes \mathcal{C} \arrow[ul, "j"']
\end{tikzcd}$$

now j is a colim-preserving functor, so, by general stuff, it has a right adjoint, $j^!$.

Set $F(W) := i \circ j^!$

$F(W)$ is not a tensor functor.

ex) X is a spc. $\quad \begin{array}{c} Sh_{ch}(X) \\ f_* \big\downarrow \big\uparrow \Gamma \\ Sh_{ch}(*) = \text{ch cplx} \end{array}$ $\quad X \xrightarrow{f} *$

Γ adjoint of f

Γ doesn't preserve $\otimes$: $\Gamma(K \otimes K') \not\cong \Gamma(K) \otimes \Gamma(K')$

Check: is F a tensor functor?

$$F(M \amalg N) = (M \amalg N) \otimes e = (M \otimes e) \otimes (N \otimes e) \quad ✓.$$

ex) $C^*(X)$-mod : $C^*(\mathrm{Map}(M \amalg N, X))$-mod $= C^*(\mathrm{Map}(M,X) \times \mathrm{Map}(N,X))$ - mod

$\overset{\text{Künneth}}{\cong} \left[C^*(\mathrm{Map}(M,X)) \otimes C^*(\mathrm{Map}(M^*,X))\right]$-mod.

$\cong C^*(\mathrm{Map}(M,X))$ -mod $\otimes$ $C^*(\mathrm{Map}(N,X))$-mod.

(this is the basic fact about $\otimes$ of k-lin cats:

$A \otimes B$-mod $\cong$ A-mod $\otimes$ B-mod)

$\Rightarrow$ defines a field theory.

Q: what was wrong w/ the $F : M \longmapsto C^*(M)$-mod

$$F(M \amalg M') \neq F(M) \otimes F(M')$$