John Francis [6] TFTs

Return to example. ex $QC_X$ ← quasicoherent sheaves on $X$ ← some space

start w/ mnflds of dim $n-1$.

$$\begin{array}{c} M \\ \downarrow \\ * \end{array} \rightsquigarrow X^* \cong X$$

$$F: M^{n-1} \longmapsto QC_{X^M} \cong M \otimes QC_X$$

using this eval. map get adjunction:

$$\begin{tikzcd} QC_{X^M} \arrow[d, shift left, "ev_*"] \\ QC_X \arrow[u, shift left, "ev^*"] \end{tikzcd} \qquad \begin{tikzcd} X \cong X^* \arrow[r] & X^M \arrow[d, "ev"] \\ & X \end{tikzcd}$$

$X = \operatorname{spec} A$

$$\begin{tikzcd} (M\otimes A)\text{-mod} \arrow[d, shift left, "\operatorname{Res}\ (*\otimes A \to M\otimes A)"] \\ A\text{-mod} \arrow[u, shift left, "\operatorname{Ind}(ev)"] \end{tikzcd}$$

so, compute $ev_* ev^*(1 = \mathcal{O}_X)$ ∈ $QC_X$

$= ev_* \mathcal{O}_{X^M}$ is commutative DGA in $QC_X$

Thus identifies $QC_{X^M} \cong \operatorname{Mod}_{ev_*\mathcal{O}_{X^M}}(QC_X)$

↑ QC sheaves on $X$ w/ an action of $ev_*\mathcal{O}_X$

$M = S^1$, $X = \operatorname{spec} A$

$S^1 \otimes A = HH_* A$

$\quad = \mathcal{O}(X^{S^1})$

$$\begin{tikzcd} S^0 \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & S^1 \end{tikzcd} \quad \overset{\text{apply } (\cdot)}{\rightsquigarrow}$$

$$\begin{tikzcd} X^{S^1} \arrow[r, "ev"] \arrow[d, "ev"] \arrow[dr, phantom, "\text{ho PB}"] & X \arrow[d] \\ X \arrow[r] & X^{S^0} \end{tikzcd} \quad \overset{\text{spec}}{\longleftrightarrow} \quad \begin{tikzcd} S^0\otimes A \cong A\otimes A \arrow[r] \arrow[d] & A \arrow[d] \\ A \arrow[r] & S^1\otimes A \end{tikzcd}$$

use push-pull formula:

$$\begin{tikzcd} X''' \arrow[r, "f'"] \arrow[d, "g'"] \arrow[dr, phantom, "PB"] & X' \arrow[d, "g"] \\ X'' \arrow[r, "f"] & Y \end{tikzcd} \longrightarrow g^* f'_* = f_* g^*$$

$\Rightarrow$ $\underline{ev_* ev^* = \Delta^*\Delta_*}$

$\Delta^*\Delta_*\mathcal{O}_X = HH_*(\mathcal{O}_X)$ ← HH-sheaf.

$\Rightarrow QC_{X^{S^1}} = \operatorname{Mod}_{\Delta^*\Delta_*\mathcal{O}_X}(QC_X) \cong \operatorname{Mod}_{HH_*(X)}(QC_X)$.

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Cases: $\emptyset \xrightarrow{D^n} \emptyset^{n-1}$ ← empty $n-1$ mnfld.

$S^n \quad D$

$S^{n-1} \circlearrowright D^n \searrow \emptyset^{n-1}$

$$\begin{tikzcd} & X^{D^n} \arrow[dl, "\sigma"'] \arrow[dr] & \\ X^{S^{n-1}} & & X^{\emptyset} = * \ (= \operatorname{spec} k) \end{tikzcd}$$

So, $D^n$ yields:

$$\begin{tikzcd} & QC_X \arrow[dr, "\Gamma"] & \\ QC_{X^{S^{n-1}}} \arrow[ur, "s^*"] \arrow[rr, dashed] & & QC_* = k\text{-mod} \end{tikzcd}$$

derived global sections.

$$\mathcal{F}\left( \underset{S^{n-1} \to \emptyset}{D^n} \right) \cong \underbrace{\Gamma \circ s^*}_{\text{a functor}}$$

Note: "Thm" $X, Y$ nice algebraic spaces

$\exists$ product functor: $QC_{X \times Y} \xrightarrow{\sim} \operatorname{Fun}(QC_X, QC_Y)$ [see Toën]

$$\begin{tikzcd} X \times Y \arrow[r, "\pi_Y"] \arrow[d, "\pi_X"'] & Y \arrow[d] \\ X \arrow[r] & * \end{tikzcd}$$

$\pi_{Y*}(M \otimes \pi_X^* -)$

So, via the above thm, $\Gamma \circ s^*$ corresps to a sheaf on $X^{S^{n-1}}$.

$$\begin{aligned} QC_{X^{S^{n-1}}} &\xrightarrow{\sim} \operatorname{Fun}(QC_{X^{S^{n-1}}}, k\text{-mod}) \\ M &\longmapsto \Gamma(M \otimes -) \end{aligned}$$

find $M$ s.t.

$$\Gamma(M \otimes -) \simeq \Gamma \circ s^*(-)$$

$$\Gamma_{X^{S^{n-1}}}(M \otimes -) = \Gamma_X s_*(M \otimes -)$$

answer:

$$M = s_* \mathcal{O}_X$$

$$\Gamma_{X^{S^{n-1}}}(s_*\mathcal{O}_X \otimes (-)) = \Gamma s_*(\mathcal{O}_X \otimes s^*(-)) = \Gamma_X s_* s^*(-) = \Gamma_{X^{S^{n-1}}} s_* s^*(-)$$

projection formula

$$\left( \begin{tikzcd} X \arrow[d, "f"] & M \arrow[l, hook'] \\ Y & N \arrow[l, hook'] \end{tikzcd} \qquad f_* M \otimes N \cong f_*(M \otimes f^* N) \right)$$

under this correspondence,

$$\underline{\Gamma \circ s^* \cong s_* \mathcal{O}_X}$$

Philosophy: what happens on disks should determine the theory.

$$\mathcal{F}\left( \underset{\emptyset^{n-1} - \emptyset^{n-1}}{S^n} \right) \in \operatorname{Fun}(k\text{-mod}, k\text{-mod})$$

$$\begin{tikzcd} & X^{S^n} \arrow[dl, "i"'] \arrow[dr, "j"] & \\ * & & * \end{tikzcd}$$

Case: $X = \operatorname{Spec} A$, $Y = \operatorname{Spec} B$

$$\begin{aligned} A \otimes B\text{-mod} &\xrightarrow{\sim} \operatorname{Fun}(A\text{-mod}, B\text{-mod}) \\ M &\longmapsto M \otimes (-) \end{aligned}$$

In this case $j_* i^*(-) = j_* i^*(k) \otimes - = (\Gamma \mathcal{O}_{X^{S^n}}) \otimes -$ sheaf cohomology of $X^{S^n}$

try to push further: $\mathcal{F}(M) \equiv \Gamma(\mathcal{O}_X^m)$