John Francis $\boxed{\text{II}}$ TQFT's

Pontrjagin 30's wanted to compute $\pi_i S^n$

known:

- $\pi_i S^n = 0 \quad i<n$ (by cellular approx.)
- $\pi_n S^n = \mathbb{Z}$ (by degree)

Pontrjagin: suppose $S^{n+k} \xrightarrow[f \text{ smooth}]{} S^n \ni x$ regular value

$\Rightarrow f^{-1}(x)$ is a mnfld.

given a homotopy btwn regular values:

$$\begin{tikzcd}
M_x =: f^{-1}(0) \arrow[r, hook] \arrow[dr] & W = f^{-1}[0,1] \arrow[d] & f^{-1}\{1\} =: M_{x'} \arrow[l, hook'] \arrow[dl] & \{0\} \arrow[r, hook] \arrow[dr, "x"] & {[0,1]} \arrow[d] & \{1\} \arrow[l, hook'] \arrow[dl, "x'"] \\
& S^{n+k} \arrow[rrr, "f"'] & & & S^n &
\end{tikzcd}$$

W is a mnfld w/ bdy $\partial W = M_x \amalg M_{x'}$

extra structure: $S^{n+k} \xrightarrow{f} S^n \ni x$ comes w/ an embedding into a sphere.

pick ε tubular nbhd of x, D_ε

$f^{-1} D_\varepsilon \cong \nu_f =$ normal bundle of the embedding.

$\exists$ a natural trivialization of $\nu_f \cong f^{-1}(D_\varepsilon) \times \mathbb{R}^n$

$= M_x \times \mathbb{R}^n$

M_x is a submanifold of S^{n+k} w/ a trivialization of the normal bundle ν_f. (an $n+k$-framing)

Pontrjagin proved:

$$\boxed{\pi_{n+k} S^n \cong \Omega_k^{\text{framed in } \mathbb{R}^{n+k}} \underset{\substack{\text{cobordism}\\ \text{classes}}}{=} \left(\begin{array}{l} k\text{-mnflds w/ framing on normal bundle} \\ \text{embedded in } \mathbb{R}^{n+k} \end{array} \right)}$$

exercise find a k-manifold which doesn't occur in Ω_k^{n+k}

(really most of them!)

fact: LHS is actually easier to compute.

- want to modify RHS to get rid of the framing requirement

Thom (1950's) (Thesis)

Defn "extra structure" is a sequence of spaces

$$\left\{ \begin{array}{c} B_n \\ \downarrow f_n \\ BO(n) \end{array} \right\} \quad \text{and maps} \quad \left\{ \begin{tikzcd} B_n \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & B_{n+1} \arrow[d] \\ BO(n) \arrow[r] & BO(n+1) \end{tikzcd} \right\}$$

aside on classifying spaces : For G a top. grp.
then $\exists$ principal G-bundle $EG \longrightarrow BG$
contractible

given X, $[X, BG] \cong \{G\text{-bundles on } X\} /$ equivalence of bundles
bijection of sets

given $V \to M$

$\boxed{\text{for}}$ $O(n)$, $BO(n) \cong Gr_n(\mathbb{R}^\infty)$, $EO(n)$ is the tautological bundle

Then, a B-structure on V is a lift $\tilde{V}$ to

$$\begin{tikzcd} V \arrow[d] & B_n \arrow[d, "f_n"] \\ M \arrow[r] \arrow[ur, dashed, "?"] & BO(n) \end{tikzcd}$$

ex) (a) $B_n = BSO(n) \simeq \widetilde{Gr}_n(\mathbb{R}^\infty) :=$ oriented n-planes in $\mathbb{R}^\infty$

$$\begin{tikzcd} V \arrow[d] & BSO(n) \arrow[d] \\ M \arrow[r, "V"'] \arrow[ur, dashed] & BO(n) \end{tikzcd}$$

then homotopy classes of lifts are equiv. to orientations of V.

(b) $B_n = BSpin(n)$

$$\begin{tikzcd} BSpin(n) \arrow[d, "2\times \text{ cover}"] \\ B(SO(n)) \end{tikzcd}$$

$$\left\{ \begin{tikzcd} V \arrow[d] & BSpin(n) \arrow[d] \\ M \arrow[r, "V"'] \arrow[ur, dashed] & BSO(n) \end{tikzcd} \right\}_{/ho} = \left\{ \begin{array}{c} \text{spin structures} \\ \text{on } V \\ \text{compatible w/} \\ \text{an orientation on } V \end{array} \right\}_{/equ}$$

(c) cmplx:

$$\begin{tikzcd} U(n) \arrow[d] \\ O(2n) \end{tikzcd} \qquad \begin{tikzcd} BU(n) \arrow[d] & \simeq & Gr_n(\mathbb{C}^\infty) \arrow[d] \\ BO(n) & \equiv & Gr_{2n}(\mathbb{R}^\infty) \end{tikzcd}$$

$$\left\{ \begin{tikzcd} & BU(n) \arrow[d] \\ M \arrow[r, "V"'] \arrow[ur, dashed] & BO(2n) \end{tikzcd} \right\}_{/homotopy} \simeq \{ \text{(almost) cmplx. structures on } V \}_{/equiv.}$$

(d) $B_n = *$

$$\begin{tikzcd} B_n = * \arrow[d] \\ BO(n) \end{tikzcd}$$

given $\left\{ \begin{tikzcd} & * \simeq B_n \arrow[d] \\ M \arrow[r, "V"'] \arrow[ur, dashed] & BO(n) \end{tikzcd} \right\}_{/ho} \simeq \left\{ \text{framings of } \begin{array}{c} V \\ \downarrow \\ M \end{array} \right\}_{/equiv.}$

$$\begin{tikzcd} V \arrow[dr] & \simeq & M \times \mathbb{R}^n \arrow[dl] \\ & M & \end{tikzcd}$$

<u>cobordism</u> w/ <u>extra structure</u>

Defn Let $\{\overset{B}{\overset{\shortparallel}{B_i}}, f_i\}$ be extra structure.

$\Omega_n^B :=$ cobordism of n-mflds w/ <u>stable</u> B-structure.

$= \left\{ \begin{array}{l} \text{closed, cpt.} \\ n\text{-mflds} \end{array} \text{ w/ embedding } M^n \xrightarrow{f} \mathbb{R}^{n+k}, \ \underset{\shortparallel}{B_k}\text{-structure on } \nu_f \text{ normal bundle} \right\}$ / cobordism w/ stable equivalence

$$\left(\begin{tikzcd} & B_k \arrow[d] \\ M \arrow[r, "\nu_f"'] \arrow[ur, dashed] & BO(k) \end{tikzcd} \text{ lift} \right)$$

cobordism w/ stable equivalence :

define $M^n \sim 0$ if $\exists$ an $(n+1)$-mnfld W w/

$$\begin{tikzcd} W \arrow[r, hook, "\iota_2"] & \mathbb{R}^{n+k+1} \\ \partial W \arrow[u, hook] \arrow[r, hook] & \mathbb{R}^{n+k} \arrow[u, hook] \\ M \arrow[u, phantom, "\cong" sloped] \arrow[ur, hook, "\iota"'] & \end{tikzcd}$$

$\partial W \cong M$ as mnflds
and ν_2 has a B_k-struct. that restricts to the B_k-struct on ν_ι

i.e.

$$\begin{tikzcd} M \arrow[rr, dashed] \arrow[d, hook] \arrow[drr, "\nu_\iota"] & & B_k \arrow[d] \\ W \arrow[urr, dashed] \arrow[rr, "\nu_2"'] & & BO(k) \end{tikzcd}$$

must be homotopy equivalent,

clarification : $\begin{array}{c} V \\ \downarrow \\ M \end{array}$ VB of rk k a B_k-structure is a lift

$$\begin{tikzcd} & B_k \arrow[d] \\ M \arrow[ur, dashed] \arrow[r, "V"'] & BO(k) \end{tikzcd}$$

and a stable B_k-struct. is an equivalence class

$g \sim g'$

$$\begin{tikzcd} & B_{k+m} \arrow[d] \\ M \arrow[ur, dashed, "g"] \arrow[r, "V \oplus \mathbb{R}^m"'] & BO(k+m) \end{tikzcd} \qquad \begin{tikzcd} & B_{k+m'} \arrow[d] \\ M \arrow[ur, dashed, "g'"] \arrow[r, "V \oplus \mathbb{R}^{m'}"'] & BO(k+m') \end{tikzcd}$$

extend to

$$\begin{tikzcd} B_{k+m} \arrow[r, "\varphi"] \arrow[d] & B_{k+m''} \arrow[d] \\ BO(k+m) \arrow[r, "\varphi"] & BO(k+m'') \\ B_{k+m'} \arrow[uur] \arrow[d] & \\ BO(k+m') \arrow[uur] & \end{tikzcd}$$

$\varphi \circ g \underset{ho}{\simeq} \varphi \circ g'$

$\Rightarrow g \sim g'$

Theorem (Thom) $\pi_n MB \cong \Omega_n^B$

$\uparrow$ Thom spectrum of $\{B_n\}$

Defn **spectrum** is a sequence of spaces $(X_n, *)$ w/ maps

$\varphi_n : \Sigma X_n \to X_{n+1}$ (can always massage to an Ω-spectrum)

These define a generalized cohomology theory.

ex) $X(n) = K(\mathbb{Z}, n)$ (know $\Omega K(\mathbb{Z}, n+1) \underset{ho}{\simeq} K(\mathbb{Z}, n)$)

take adjoint (hurst)

corresp to $[X, X_n] \cong H^n(X, \mathbb{Z})$

in general write $\underline{X^n(X)}$.

Also defines a homology theory

(2) $X_n = S^n$ $\quad \Sigma S^n \underset{homeo}{\cong} S^{n+1}$ give a sphere spectrum

defines a homology theory via:

$$H_n(X) = \lim_{k \to \infty} \pi_{k+n}(X) =: \pi_n^S(X).$$

(3) $X_n = MB_n$ = Thom spectrum w/ B structure

$$X \rightsquigarrow \Omega_n^B(X) := \left\{ \begin{array}{l} \text{closed, cpct } n\text{-mnflds } M \text{ w/ stable structure } B \text{ on } M \underset{\nu}{\hookrightarrow} \mathbb{R}^{n+k} \\ \text{w/ a map } M \xrightarrow{f} X \end{array} \right\} \Big/ \text{cobordism}$$

$(M, f) \sim 0$ if $\exists\, W^{n+1}$ w/ $\tilde{f} : W^{n+1} \to X$ s.t. $\partial W^{n+1} \cong M$ and $\tilde{f}|_{\partial W} = f$.

i.e.

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, phantom, "\cong" sloped] & X \\ \partial W \arrow[d, hook] & \\ W \arrow[uur, dashed, "\tilde{f}"'] & \end{tikzcd}$$

note: $\Omega^B_n(pt) = \Omega^B_n$ defines a homology theory.