

John Francis □ T&T'S

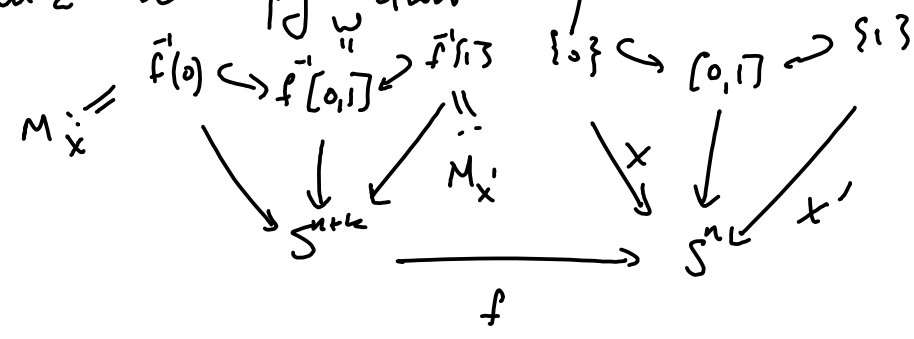
Pontrjagin 30's wanted to compute $\pi_i S^n$

- known:
- $\pi_i S^n = 0$ $i < n$ (by cellular approx.)
 - $\pi_n S^n = \mathbb{Z}$ (by degree)

Pontrjagin: suppose $S^{n+k} \xrightarrow{f} S^n$ f smooth $\ni x$ regular value

$\Rightarrow f^{-1}(x)$ is a manifold.

given a homotopy between regular values:



W is a manifold w/ $\partial W = M_x \amalg M_{x'}$

extra structure: $S^{n+k} \xrightarrow{f} S^n \ni x$ comes w/ an embedding into a sphere.

pick ε tubular nbhd of x , D_ε

$f^{-1} D_\varepsilon \cong \nu_f =$ normal bundle of the embedding.

\exists a natural trivialization of $\nu_f \cong f^{-1}(D_\varepsilon) \times \mathbb{R}^n = M_x \times \mathbb{R}^n$

M_x is a submanifold of S^{n+k} w/ a trivialization of the normal bundle ν_f . (an $n+k$ -framing)

Pontrjagin proved:

$$\prod_{n \in \mathbb{N}} S^n \cong \Omega_k^{\text{framed in } \mathbb{R}^{n+k}} = \text{cobordism classes of } k\text{-manifolds w/ framing or normal bundle}$$

exercise find a k -manifold which doesn't occur in Ω_k^{n+k}

(really most of them!)

fact: LHS is actually easier to compute.

want to modify RHS to get rid of the framing requirement

Thom (1950's) (Thesis)

Defn "extra structure" is a sequence of spaces

$$\left\{ \begin{array}{c} B_n \\ \downarrow f_n \\ BO(n) \end{array} \right\} \text{ and maps } \left\{ \begin{array}{c} B_n \rightarrow B_{n+1} \\ \downarrow \cong \downarrow \\ BO(n) \rightarrow BO(n+1) \end{array} \right\}$$

aside on classifying spaces: For $G \subset \text{top. grp.}$
 then \exists principal G -bundle $EG \rightarrow BG$
 given X , $[X, BG] \cong \{G\text{-bundles on } X\} / \text{equivalence of bundles}$
 ↑
 bijective of sets
 (EG is contractible)

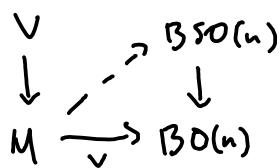
$$\begin{array}{c} \text{given } V \\ \downarrow \\ M \end{array}$$

[for $O(n)$, $BO(n) \subseteq Gr_n(\mathbb{R}^\infty)$, $EO(n)$ is the total space bundle]

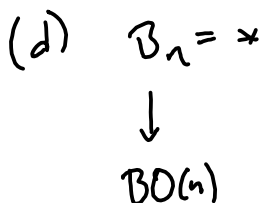
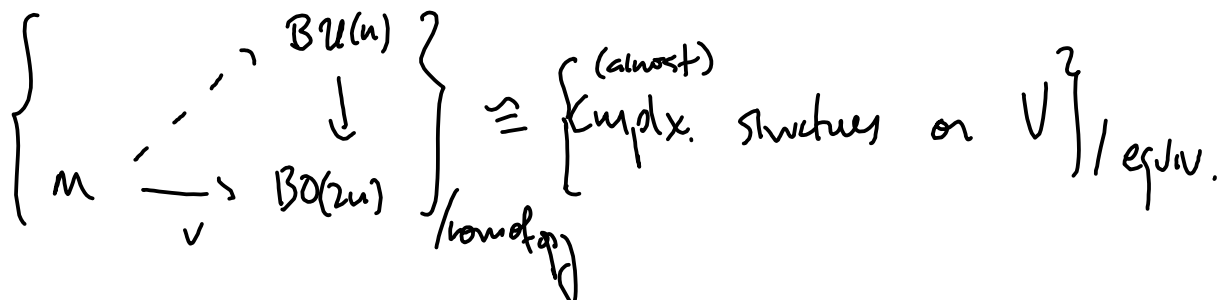
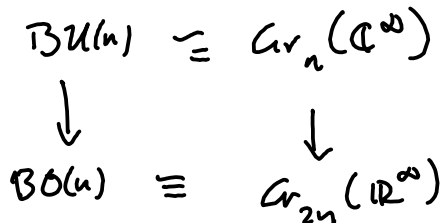
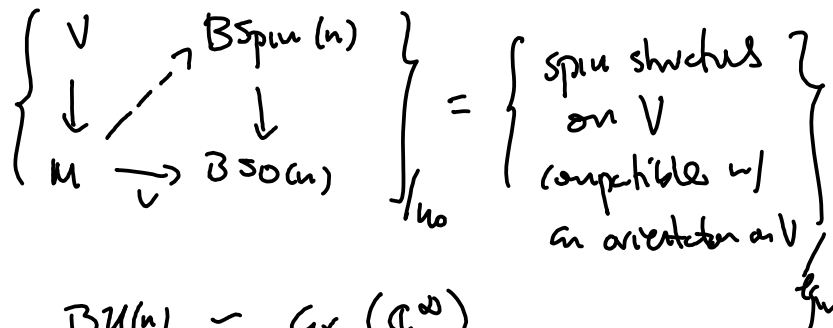
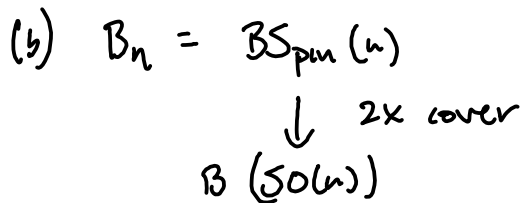
Then, a B-structure on V is a lift \tilde{V} to

$$\begin{array}{ccc} V & & B_n \\ \downarrow & \dashrightarrow & \downarrow f_n \\ M & \rightarrow & BO(n) \end{array}$$

ex) (a) $B_n = BSO(n) \cong \widetilde{Gr}_n(\mathbb{R}^\infty) :=$ oriented n -planes in \mathbb{R}^∞



then homotopy classes of lifts are equiv. to orientations of V .



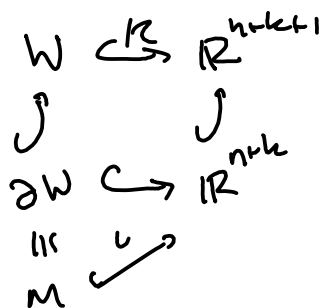
Cobordism w/ extra structure

Defn let $\{B_i, f_i\}$ be extra structure.

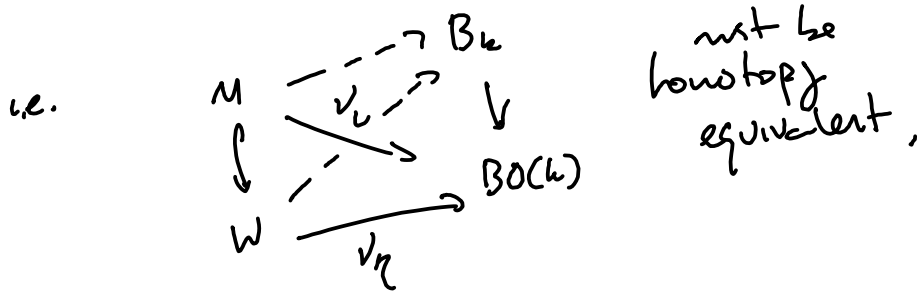
$\Omega_n^B :=$ cobordism of n -manifolds w/ stable B -structure.
 $= \left\{ \begin{array}{l} \text{closed, cpt.} \\ n\text{-manifolds} \end{array} \right\}$ w/ embedding $M \xrightarrow{f} \mathbb{R}^{n+k}$, B_k -structure on V mod $\{ \text{stable equivalence} \}$
 $\left(\begin{array}{ccc} & B_k & \\ & \downarrow & \\ M & \xrightarrow{\nu} & BO(n) \end{array} \right)$ iff

Cobordism w/ stable equivalence :

define $M^1 = 0$ if \exists an $(n+1)$ -manifold W w/

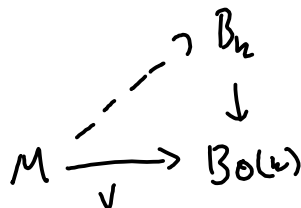


$\partial W \cong M$ as manifolds
and ν has a B_k -struct. that restricts to the B_k -struct on $\nu|_M$

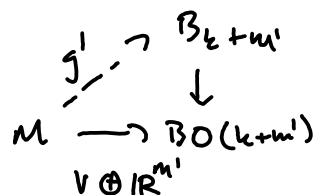
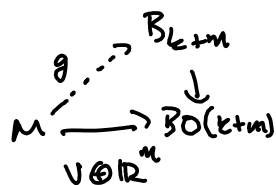


clarification :

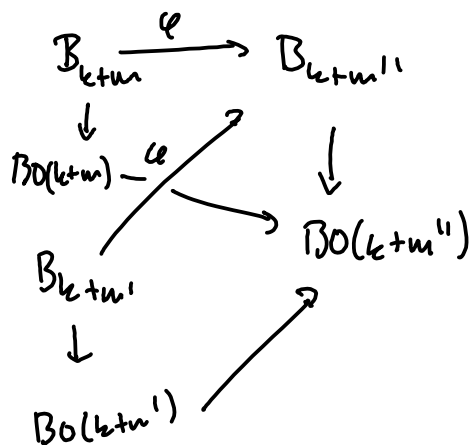
ν is a lift of $\nu|_M$ of rank k a B_k -structure is a lift



and a stable B_k -struct. is an equivalence class $g \sim g'$



extend to



$$\begin{array}{c}
 \varphi \circ g \sim_{ho} \varphi \circ g' \\
 \hline
 \Rightarrow g \sim g'
 \end{array}$$

Theorem (Thom) $\Pi_n MB \cong \Omega_n^B$

↑
Thom spectrum of $\{B_n\}$

Defn spectrum is a sequence of spaces $(X_n, *)$ w/ maps

$\varphi_n: \Sigma X_n \rightarrow X_{n+1}$ (can always massage to an Ω -spectrum)

These define a generalized cohomology theory.

ex) $X(n) = K(\mathbb{Z}, n)$ (know $\Omega K(\mathbb{Z}, n+1) \xrightarrow[\cong]{\simeq} K(\mathbb{Z}, n)$)
take adjoint (hard)

corresp to $[X, X_n] \cong H^n(X, \mathbb{Z})$

in general write $X^n(X)$.

Also
 defines
 a homology
 theory

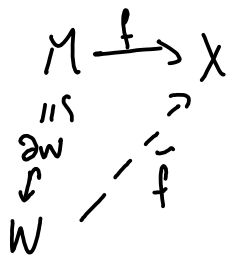
(2) $X_n = S^n$ $\Sigma S^n \xrightarrow[\cong]{\simeq} S^{n+1}$ give a sphere spectrum

defines a homology theory via:

$H_n(X) = \lim_{k \rightarrow \infty} \Pi_{k+n} (X) =: \Pi_n^S(X)$.

(3) $X_n = MB_n =$ Thom spectrum w/ B structure

$X \mapsto \Omega_n^B(X) := \left\{ \begin{array}{l} \text{closed, cpt n-manifolds } M \text{ w/ stable structure } \\ \text{w/ a map } M \xrightarrow{f} X \end{array} \right\}$
 $\left. \begin{array}{l} \text{B or} \\ M \xrightarrow{\nu} \mathbb{R}^{n+k} \end{array} \right\}$
 /cobordism



i.e.

$(M, f) \sim 0$ if $\exists W^{nh}$ w $\tilde{f} : W \rightarrow X$
 s.t. $\partial W^{nh} \cong M$ and $\tilde{f}|_{\partial W} = f$.

note

$$\Omega_n^B(pt) = \Omega_n^B$$

defines a homology theory.