John Francis: $\boxed{2}$

Defn of Thom Space: $\begin{array}{c} V \\ \downarrow \\ X \end{array}$ VB then $Th(V) = Disk(V)/_{S}$

$Disk(V)$ = disk bundle of V

= V^+ 1-pt. compactification of V if X cpct.

note: V is trivial $= \mathbb{R}^n \times V$.

Then $Th(V) \cong \Sigma^n X_+$ ← disjoint union w/ basept $+$.

If V any VB, $Th(V \oplus \mathbb{R}^n_X) \cong \Sigma^n Th(V)_+$ ← basept. $+$ at ∞

ex) Let V be the VB which is $\begin{array}{c} \gamma^n \\ \downarrow \\ Gr_n(\mathbb{R}^\infty) \end{array}$ ← universal n-plane bundle $= EO(n)$.

Take $Th(\gamma^n)$; forms a spectrum:

$$\begin{array}{ccc} Gr_n(\mathbb{R}^k) & \hookrightarrow & Gr_{n+1}(\mathbb{R}^{k+1}) \\ \downarrow & & \downarrow \\ Gr_n(\mathbb{R}^\infty) & \to & Gr_{n+1}(\mathbb{R}^\infty) \end{array}$$

(add particular 1-dml space to each k-space)

Obs: γ^{n+1} pulls back to $\gamma^n \oplus \underline{\mathbb{R}}$ over this inclusion.

have a map $\gamma^n \oplus \underline{\mathbb{R}} \longrightarrow \gamma^{n+1}$ (of VB's) preserves length of vectors.

$\Rightarrow$ induces map on Thom spaces

$$\begin{array}{c} Th(\gamma^n \oplus \underline{\mathbb{R}}) \to Th(\gamma^{n+1}) \\ \text{\rotatebox{90}{$\cong$}} \\ \Sigma Th(\gamma^n) \end{array}$$

Defn: Let $MO_n := Th\left(\begin{array}{c} \gamma^n \\ \downarrow \\ Gr_n(\mathbb{R}^\infty) \end{array}\right)$ w/ structure maps ex) $B_n = BO_n \qquad MB_n = MO_n$.

Observe:

$$\begin{tikzcd} f_n^{-1}\gamma^n \arrow[r, dash] \arrow[d] & B_n \arrow[r, "\varphi_n"] \arrow[d, "f_n"] & B_{n+1} \arrow[d] & f_{n+1}^{-1}\gamma^{n+1} \arrow[l] \\ \gamma^n \arrow[r, dash] & BO(n) \arrow[r] & BO(n+1) & \end{tikzcd}$$

$$\varphi_n^{-1} f_{n+1}^{-1}\gamma^{n+1} \cong f_n^{-1}\gamma^n \oplus \underline{\mathbb{R}}$$

by same argument,

$$\Sigma Th(f_n^{-1}\gamma^n) \to Th(f_{n+1}^{-1}\gamma^{n+1})$$

This data gives MB w/

$MB_n := Th(f_n^{-1}\gamma^n)$ w/ structure maps

Defn: π_*(Spectrum): let X be a spectrum.

$$\pi_n X := \varinjlim_{i\to\infty} \pi_{n+i} X_i$$

ex) Let $X = \$:=$ sphere spectrum.

$X_n = S^n \quad \pi_n \$ = \lim_{i\to\infty} \pi_{n+i} S^i$ (Freudenthal suspension thm says sequence stabilizes for $i > n$)

[How does a spectrum define a homology thry?:

E a spectrum, then E_* is a homology thry:

$$E_* X := \lim_{k\to\infty} \pi_{*+k}(E_k \wedge X_+)$$
]

immediately observe: $\pi_* E = E_*(pt)$.

Now Thom's Thm says: $\pi_n MB \cong \Omega_n^B$

<u>More general result</u> : (Thom)

$$\Omega_n^B(K) \cong (MB)_n K$$

K — any top. spc.

$\Omega_n^B(K)$ — (cobordism grp. of n-mnflds w/ target K.)

$$(MB)_n K = \lim_{i\to\infty} \pi_{n+i} MB_i \wedge K_+$$

We will only deal w/ $K = *$ pt.

construct maps

"(P-T construction)": $\lim_{i\to\infty} \pi_{n+i} MB_i \dashleftarrow \Omega_n^B$

Let M be an n-mnfld w/ struct. B

$M^n \xhookrightarrow{f} \mathbb{R}^{n+k}$ $\qquad \nu_f \to M$ normal bundle of embedding

w/ lift
$$\begin{tikzcd} & B_k \arrow[d] \\ M \arrow[ur, dashed, "\tilde{\nu}_f"] \arrow[r, "\nu_f"'] & BO(k) \end{tikzcd}.$$

Take an ε-nbhd of M in $\mathbb{R}^{n+k}$

D_ε

$$\begin{tikzcd} M \arrow[r, hook, "f"] & \mathbb{R}^{n+k} & \\ \nu_f \cong D_\varepsilon \arrow[u] \arrow[r] & \mathbb{R}^{n+k} \times \mathbb{R}^{n+k} \arrow[u] \arrow[r, "+"] & \mathbb{R}^{n+k} \end{tikzcd}$$

Choose ε small enough so that the composite is an embedding.

$\mathbb{R}^{n+k}$ sits inside its 1-pt. compactification $\quad \mathbb{R}^{n+k} \subset S^{n+k}$

$$\nu_f \xhookrightarrow{i} \mathbb{R}^{n+k}$$
$$Th(\nu_f) \leftarrow S^{n+k} \ni p$$

$$p \in S^{n+k} \longmapsto \begin{cases} j^{-1}(p) & \text{if } p \in \mathbb{R}^{n+k} \\ * & \text{if } p = \infty \end{cases}$$ "P-T collapse map"

and,
$$\begin{tikzcd} \nu_f \arrow[r] \arrow[d] \arrow[dr, phantom, "PB"] & f_k^{-1}\gamma^k \arrow[d] \\ M \arrow[r] & B_k \end{tikzcd}$$
yields $Th(\nu_f) \to Th(f_k^{-1}B_k) =: MB_k$

Then, compose these maps:

$$S^{n+k} \to Th(\nu_f) \to Th(f_k^{-1}\gamma^k) =: MB_k$$

gives a well-defined elt. of $\underline{\pi_n MB}$.

__Check__: • only depends on B-cobordism class

• suppose $M = \partial W$ consider $M^{\times[0,1)}$ embedded in W parametrized by $[0,1)$ to yield a null homotopy

Then directions

$[\alpha] \in \pi_n(MB)$

$$\begin{tikzcd} S^{n+k} \arrow[r, "\alpha"'] & MB_k \arrow[r] & MO_k = Th(\gamma^k \to Gr_k \mathbb{R}^\infty) \\ & & Gr_k(\mathbb{R}^\infty) \arrow[u, hook, "\text{zero section}"'] \end{tikzcd}$$

(away from ∞ is a mnfld)

deform $Th(f_n) \circ \alpha$ to make S^{n+k} transverse to $Gr_k(\mathbb{R}^\infty)$.

__Define__ $M^n = (Th(f_n) \circ \alpha)^{-1}(Gr_k \mathbb{R}^\infty)$

check has dimension n (and is a mnfld)

Using some lifting arguments, M^n has a B-structure. $\blacksquare$

(understanding this in terms of moduli of manifolds is worked out in the work of Galatius–Tillmann et al.)

<u>What good is this?</u>

now define E-valued cobordism invariants:

<u>General recipe:</u> let E be a spectrum, determining a

w/ $MB \longrightarrow E$

a map of spectra (i.e. nat'l transf. of homotopy thy.

$MB_n \xrightarrow{h_n} E_n \quad \forall n$

$$\begin{array}{ccc} \Sigma MB_n & \longrightarrow & \Sigma E_n \\ \downarrow & \circlearrowleft & \downarrow \\ MB_{n+1} & \longrightarrow & E_{n+1} \end{array}$$
)

then can produce cobordism invrnts of B-mnflds $E_*(pt)$.

<u>2 inputs</u>: $p \in E^*(B)$, $\alpha_*[M] \in E_*(M)$ $\qquad B = \varinjlim_{n\to\infty} B_n$

<u>Fundamental class</u> $[M]$: $\Omega^B_*(K)$ is defined for any sp. K but if $K = M^n$ is a B-mnfld. have

$$M \xrightarrow{Id} M \qquad (M^n, Id) \in \Omega^B_n(M)$$
$$\| $$
$$[M] \quad \text{fund. class.}$$

get: $\Omega^B_*(M) \xrightarrow{\alpha_*} E_*(M) \ni \alpha_*[M]$. $\qquad p \in E^*(B)$

$$M \xrightarrow{f} B_k \longrightarrow B$$

$f^*p \in E^*(M)$

Now have pairing : $E^*(M) \times E_*(M) \xrightarrow[\langle\, , \rangle]{} E_*(pt)$

$$f^* p \qquad \alpha_*[M] \longrightarrow \langle f^* p, \alpha_*[M] \rangle$$

is a B-cobordism invariant.

<u>why is it a cobordism invariant?</u>

suffices to show $p(M) = 0$ if $M \cong \partial W$ $\big(\partial W \to W \to (W, \partial W)\big)$

B-structure.

assume $\exists$ such a W $\quad [W, \partial W] \in \Omega^B_{n+1}(W, \partial W) \cong \tilde{\Omega}^B_{n+1}(W/\partial W)$

apply homology $\qquad$ rel. fund. class.

$$\cdots \to \Omega^B_*(\partial W) \longrightarrow \Omega^B_*(W) \xrightarrow[\delta]{} \Omega^B_*(W, \partial W) \to \cdots \qquad \text{LES}$$

<u>note</u> $\quad \delta[W, \partial W] = [\partial W]$.

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, hook, "\partial W"'] \arrow[dr, phantom, "\circlearrowleft"] & B_k \\ W \arrow[ur, "\tilde{f}"'] & \end{tikzcd}$$

(+ compatibility w/ B-struct)

$f = \tilde{f} \circ \partial W \qquad$ so, write

$$p(M) = \langle f^* p, \alpha_*[M] \rangle = \langle \partial W^* \circ \tilde{f}^* p, \alpha_* \delta[W, \partial W] \rangle$$

by naturality of $\langle\, , \rangle$ $\quad = \langle \tilde{f}^* p, \partial W_* \alpha_* \delta[W, \partial W] \rangle$

$$= \langle \tilde{f}^* p, \alpha_* \partial W_* \delta[W, \partial W] \rangle$$

but $\partial W_* \circ \delta = 0$ from LES

$\Rightarrow \qquad \underline{= 0}$ $\blacksquare$

ex) $E =$ Eilenberg-MacLane spaces.

$H^*X \times H_*X \xrightarrow[\langle , \rangle]{} \mathbb{Z}$ is the usual pairing.

need map $MB \overset{- - -\to}{\alpha} H$ set $B = BSO$

then, $MB := MSO$ $\exists$ map $MSO \xrightarrow[\alpha]{} E$

$\pi_0 MSO \cong \mathbb{Z}$, α exists as a consequence.

(Form of Hurewicz thm)

(note $\exists$ map $MO \to E_{\mathbb{Z}/2} = K(\mathbb{Z}/2, *)$
$\pi_0 MO = \mathbb{Z}/2$)

$$\boxed{H^*(BSO)/\text{torsion} \cong \mathbb{Z}[p_1, \dots]}$$

choose $p \in H^* BSO$ $p = \prod_{i \in I} p_i$

$p(M) = \langle f^* p, \alpha_* [M] \rangle = \prod$ of Pontrjagin #'s of the stable normal bundle.

$\alpha_*[M]$: usual $[M]$

ex) find lots of manifolds M s.t. $M \not\simeq 0$.

note if we use MO get Stiefel-Whitney numbers.

(note stable normal bundle is stably equiv to tangent bundle so P-#'s are same here as usual)