Jonathan Block [1]   2 schools of NCG: (of several)

1) Connes   Operator algebras   Today
2) Kontsevich   Categories   Tomorrow   } Friday

Beginning: Famous thm. of Gelfand, Naimark
$\exists$ equivalence of cats. btwn.

cpct. top spaces $\longleftrightarrow$ commutative C^*-algebras

$$X \longmapsto C(X)$$
$$\hat{A} \longleftarrow\!\!| \; A$$

Lusztig (1971) Thesis proved Novikov's Theorem.

Signature theorem

$$H^{2k}(M,\mathbb{Q}) \times H^{2k}(M,\mathbb{Q}) \xrightarrow{\cup} H^{4k}(M,\mathbb{Q}) = \mathbb{Q}$$

sgn(M) homotopy invrnt. and

Hirzebruch: $\operatorname{sgn}(M) = \langle L(M), [M] \rangle$

characteristic class in $H^{4k}(M,\mathbb{Q})$

$\operatorname{sgn}(M) = \operatorname{Index} D$ where D is an elliptic diff'l op.
mapping $\Omega^+(M) \longrightarrow \Omega^-(M)$, $\dim \ker D - \dim \operatorname{coker} D =: \operatorname{Ind} D$.

Q: Are there other cc's which are homotopy invrnt.? (rational cc's)

If M is simply-connected, answer is no.

Novikov Conjecture $f: M \longrightarrow B\pi$ let $\alpha \in H^*(B\pi;\mathbb{Q})$
π is grp.   form $\langle L(M) \cup f^*\alpha, [M] \rangle$
"higher signature"

Conjecture: these are homotopy invariants.

Lusztig proved the conjecture when $\pi = \mathbb{Z}^n$.

Let $T = V/\Lambda$, let $\Lambda = \mathbb{Z}^n$, T is a Torus and is a $K(\Lambda,1)$

Let $T^\vee = V^\vee/\Lambda^\vee$, $\Lambda^\vee = \{\xi \in V^\vee \mid \langle \xi, \lambda\rangle \in \mathbb{Z}\}$ $\forall \lambda$

$T^\vee \equiv$ Pontrjagin dual of Λ

= irreps. of Λ = flat unitary line bundles on T.

So, $\exists$ universal line bundle $\begin{matrix} P \\ \downarrow \\ T \times T^\vee \end{matrix}$ s.t. $P|_{T\times\xi} = \xi$

Consider $f: X \to T = K(\Lambda,1)$

form $(f\times 1): X\times T^\vee \to T\times T^\vee$, $(f\times 1)^* P \to X\times T$
$$\begin{matrix} (f\times 1)^*P \\ \| \\ \tilde{P} \end{matrix}$$

Can form the family of signature ops. $D_{\tilde{P}}$ on $X\times T^\vee$ over $\xi \in T^\vee$. $D_{\tilde{P}}$ is the signature of on X w/ "values in ξ"

$\mathrm{Ind}\, D_{\tilde{P}} = (\ker D_{\tilde{P}} - \mathrm{cok}\, D_{\tilde{P}})_\xi \in K^0(T^\vee)$ which you can show is a homotopy invariant.

Families Index thm of Atiyah-Singer says:

$\langle f^*\alpha \cup L(X), [X]\rangle$ $\quad$ (X is the same as M)

Myshchenko, Fomenko, Kasparov generalized this recall, $T = V/\Lambda = B\Lambda$, $T^\vee$ = Pdual of Λ

can take $\mathbb{C}[\Lambda] \to B(\ell^2(\Lambda))$ (by natural rep on left)

then close $\mathbb{C}[\Lambda]$ inside $B(\ell^2(\Lambda))$ to get $C^*(\Lambda)$.

Pontrjagin duality: $C^*(\Lambda) \cong C(\underbrace{\hat{\Lambda}}_{\text{Pontrjagin dual.}}) \cong C(T^\vee)$

If want to apply Connes's idea to non-abelian fundamental grps then need to consider $C^*(\Gamma)$ and families index them should take place on "Spec $C^*\Gamma$."

If A is a C^*-alg. write $\hat{A}$:= space of all irred. $*$-reps on Hilb. spc.

Myshchenko, Fomenko, Kasparov generalized A-S families index thm so that it takes place using $A = C^*(\pi_1 M)$ as a parameter spc.

Assume for simplicity, Γ is torsion free:

Aside: For A a Banach algebra, define $K_*(A) := \pi_k(GL_\infty(A))$
$= \pi_k(BGL_\infty(A))$

then,

(Baum-Connes) conj: $K_*(B\Gamma) \xrightarrow{\sim} K_*(C^*\Gamma)$ isomorphism.

note: $\underbrace{K_*(B\Lambda) \longrightarrow K_*(C^*\Lambda)}_{\text{T-duality}} = K^*(T^\vee)$

What is the sense of treating $C^*\Gamma$ as functions on some space of irreducible modules over $C^*\Gamma$?

What does $C^*\Gamma$ look like? (It is a kind of moduli spc. of all unitary irreps of Γ)

Let α be a cardinal $\# \leq \aleph_0$ and let H_α be a Hilbert spc. of this cardinality. $I := \mathrm{Irrep}(\Gamma; H_\alpha)$ is a separable, metrizable, spc.

$\mathcal{U}(H_\alpha)$ acts on I. Quotient is $\widehat{C^*\Gamma} := I/\mathcal{U}(H_\alpha)$.

<u>dichotomy</u>: either,

Type 1 1) $\widehat{C^*\Gamma}^\alpha$ is very nice i.e. not too non-Hausdorff and can understand the structure of $C^*\Gamma$ by analyzing this $\widehat{C^*\Gamma}^\alpha$

Type 2 2) $I \xrightarrow{\text{proj}} I/\mathcal{U}(H_\alpha)$ $\nexists$ a Borel section of this projection.

many NC-spaces (in either context) arise from not wanting to take quotient (too early).

In both approaches think geometrically by looking at the category of modules