Jonathan Block | NCG 2 | categorical approach.

In commutative geometry pts. $\longleftrightarrow$ irreducible modules

in NCG, things are more complicated.

Many examples: principally: quotients, deformations:

Groupoids: classify objects.

ex) groups, equiv. relatn on a Set,

$$\begin{array}{c} R \subset X \times X \\ \downarrow\downarrow \\ X \end{array}$$

notation: $\mathcal{G} \overset{s}{\underset{t}{\rightrightarrows}} X$ objects  $X/\mathcal{G}$ = quotient via equiv relation.

topological groupoids: $\mathcal{G} \overset{s}{\underset{t}{\rightrightarrows}} X$  s, t, have required structure (in Category)
(diff'l, alg)

$\mathcal{G} \times_X \mathcal{G} \to \mathcal{G}$ in Category

$\mathcal{G} \mapsto \mathcal{G}$ in Category
$\gamma \mapsto \gamma^{-1}$

Defn: in NC-world: can form $C^*(\mathcal{G})$ (analogous to the grp. alg.)
in Alg-geom: can form the stack $[X/\mathcal{G}]$ an

More examples of groupoids: $G$ a grp acting on ~~top.~~ spc. $X$
have a map $G \times X \to X$: form groupoid:

$\mathcal{G}_1 = G \times X \overset{s}{\underset{r}{\rightrightarrows}}$  $s(g, x) = x$
$r(g, x) = gx$

so, $(g_1, x_1) \circ (g_2, x_2)$
if $x_1 = g_2 x_2$
$= (g_1 g_2, x_2)$

The quotient stack $[X/G] = X/G$ should be thought of as a Morita-equivalence class of groupoids.

$\mathcal{G}_2 = G \times X \times H \rightrightarrows X \times H$

This isn't a groupoid, but $\mathcal{G}_1, \mathcal{G}_2$ have the same isom. classes of objects and the same isotropy.

From some point of view $\mathcal{G}_1$ and $\mathcal{G}_2$ are equivalent.

ex) Let $X$ be mnfld., $\mathcal{U}_\alpha$ a cover of $X$ by open sets.

Define a groupoid $\mathcal{U} := \coprod_{\alpha, \beta} (\mathcal{U}_\alpha \cap \mathcal{U}_\beta) \overset{s}{\underset{r}{\rightrightarrows}} \coprod_\alpha \mathcal{U}_\alpha$

$s(x) = x \in \mathcal{U}_\beta$, $r(x) = x \in \mathcal{U}_\alpha$

also think of $X$ as a groupoid trivially $X \overset{id}{\underset{id}{\rightrightarrows}} X$.

So $[X] \cong [\mathcal{U}]$

These represent the same stack.

ex) Let $(M^n, \mathcal{F}^q)$ be a foliated mnfld.

$q =$ codimension

As an atlas, take a $q$-dimensional submnfld $T$ which intersects every leaf.

define groupoid $\mathcal{G} := \{$paths $\gamma: I \to M$ s.t. $\gamma(0) \in T$, $\gamma(1) \in T$, $\gamma \subset \mathcal{L}$ a particular leaf$\}$ $/\sim$

$\sim =$ homotopy in leaf rel. base points.

$\mathcal{G} \overset{r}{\underset{s}{\rightrightarrows}} T$ $\quad r\gamma = \gamma(1)$, $s\gamma = \gamma(0)$ $\quad$ composition obvious

Statement $[\mathcal{G}_{T_1}] = [\mathcal{G}_{T_2}]$ for any $T_1, T_2$ satisfying the above condition.

form $C^*(\mathcal{G}_T)$. A leaf $L$ of $\mathcal{F}$ gives an irreducible module of $C^*(\mathcal{G}_T)$.

$C^*(\mathcal{G}_{T_1}) \overset{M}{\simeq} C^*(\mathcal{G}_{T_2})$ : i.e. are Morita Equivalent.

[what's intrinsic is $A$-mod not $A$ itself.]

$A \rightsquigarrow M_n(A)$ $\quad$ $A, B$ Morita equivalent if $\exists$ proj. $A$-module $P$ s.t. $B \cong \mathrm{End}_A(P)$.

**Moduli problems**

A set of objects $X$ you want to classify up to isomorphism.

try: $X/$ isomorphism

- could be a nasty space
- could have isotropy:

keeps $X/\sim$ from being a fine moduli space

$$\forall \left\{\begin{array}{c} C \\ \downarrow \\ Y \end{array}\right\} \xrightarrow{\text{get}} \begin{array}{c} \mathcal{C} \\ \downarrow \\ M \end{array}$$

$\mathcal{C}$ ← univ. family

$M$ ← moduli

**Categories of modules**

Take the whole cat. of objects and look for equivalences.

**Kontsevich School of NCG**

An NC space is a triangulated cat.

example: $X$

$$X \rightsquigarrow D^b(X)$$

$D^b(X)$ ← Cat. of certain $\mathcal{O}_X$ modules

Looking at $D^b$ really only up to equivalence.

Can happen that $\exists\, X, Y$ not isom. varieties, yet $D^b(X) \cong D^b(Y)$.

when you have such an equivalence you are implicitly solving some moduli problem.

$$D^b(X) \longrightarrow D^b(Y)$$
$$\mathcal{O}_X \longrightarrow \Phi(\mathcal{O}_X)$$

ex) Mukai Duality $\quad X$ complex torus.

$Y$ space of degree $0$ line bundles on $X$

$= $ a dual torus

$$D^b(Y) \longleftrightarrow D^b(X).$$

ex) A beautiful deformation: Look at Heisenberg grp. $\quad x, y, z \in \mathbb{R}$

$$H = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \text{center } Z(H)$$

$$\begin{array}{c} C^*(H) \\ \downarrow \\ \mathbb{R} \end{array}$$

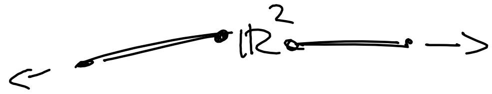
"Fourier transform at" the $z$ coord.

$$\begin{array}{ccc} A_\xi & \longrightarrow & C^*(H) \\ & & \downarrow \\ & & \mathbb{R} \end{array}$$

$$A_0 \cong C^*(\mathbb{R}^2) \cong C_0(\hat{\mathbb{R}}^2) = C_0(\mathbb{R}^2)$$

$A_{\xi \neq 0} \cong$ Compact operators on $\mathbb{R}^2$

Is a deformation which describes $\mathbb{R}^2$ deforming into a pt.




[Bott periodicity is a consequence of the deformation-invariance of this family]

$$\begin{array}{c} C(X)\otimes C^*(H) \\ \big\downarrow \\ \mathbb{R} \end{array}$$

over 0 get $X\times\mathbb{R}^2$

$\xi\neq 0$, $X\times\{pt\}$