Jonathan Block | NCG 3 Γ is a torsion free group

Baum-Connes Conj.: $K_*(B\Gamma) \xrightarrow{\mu} K_*(C_r^*\Gamma)$

μ is an isomorphism

Description of μ the assembly map:

on $B\Gamma \times \mathrm{Spec}\, C_r^*\Gamma$ $\exists$ a line bundle, the Myschenko line bundle.

$$\begin{array}{c} \mathcal{V} = E\Gamma \times_\Gamma C_r^*\Gamma \\ \downarrow \\ B\Gamma \end{array}$$ w/ fibers $C_r^*\Gamma$ as left modules.

Can think of this as a "line bundle" over $B\Gamma \times \mathrm{Spec}\, C_r^*\Gamma$

Assume $B\Gamma$ is spin^c-manifold. Then $K_*(B\Gamma) \cong K^{top *}(B\Gamma)$

$\mu : K^*(B\Gamma) \to K^*(C_r^*\Gamma)$

$$\begin{tikzcd} & B\Gamma \times \mathrm{Spec}\, C_r^*\Gamma \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ B\Gamma & & \mathrm{Spec}(C_r^*\Gamma) \end{tikzcd}$$

define μ via push-pull.

Remark: p_{2*} in K-theory is realized by taking the index of $D\!\!\!/_{\mathcal{V}}$ along fibers of p_2.

Now to get something more algebraic and more subtle.

Want a description of coherent sheaves which is global diff'l geometric,
To be able to take a lot cat of sheaves on NC spaces need to describe these objects in terms of global diff'l geometry.

Consider (for example on cmplx mnfld) the Dolbeaux alg.:

$$A = (A^{0,\bullet}X, \bar\partial)$$

Thm: $\{\text{Hol VB's on } X\} \longleftrightarrow \{C^\infty \text{ VB's w/ flat } \bar\partial\text{-connections}\}$

that is: $\bar\nabla : C^\infty(X;E) \to A^{0,1}(X;E)$

$$\bar\nabla(sf) = \bar\nabla(s)f + s\bar\partial f$$

It's holomorphic if $\bar\nabla^2 = 0$.

Let $A = (A^\bullet, d, c)$ curved DGA $(d^2 = [c, \cdot])$

Define a DG-cat. P_A consisting of objects:

$(E^\bullet, \mathbb{E})$ $\quad E^\bullet$ f.g., $\mathbb{Z}$-graded, proj. A^0-module

$\mathbb{E} : E^\bullet \otimes_{A^0} A^\bullet \circlearrowleft$ a total degree 1 homomorphism

satisfying $\mathbb{E}(e\alpha) = \mathbb{E}(e)\alpha + (-1)^{|e|} e \otimes d\alpha$

curvature: $\mathbb{E}^2(e) = -e \otimes c$

Morphisms: $\phi : (E^\bullet, \mathbb{E}) \to (F, \mathbb{F})$ of degree k

if $\phi : E^\bullet \otimes_{A^0} A \to F^\bullet \otimes_{A^0} A$ is of total degree k

$$d\varphi := \mathbb{F} \circ \varphi - (-1)^{|\varphi|} \varphi \circ \mathbb{E}, \quad (d^2 = 0)$$

$\therefore$ P_A becomes a DG-cat.

<u>Fact</u>: $Ho\,\mathcal{P}_A$ is triangulated.

<u>Thm</u>: If $A = (A^{0,\bullet}X, \bar{\partial}, 0)$, then $Ho\,\mathcal{P}_A \cong D^b$(shvs w/ coherent cohomology X)

<u>examples</u>: If $(\mathcal{E}^\bullet, \delta)$ is a cmplx of holomorphic VB's

define $(E^\bullet, \mathbb{E})$: $E^\bullet = C^\infty(X, \mathcal{E}^\bullet)$, $\bar{\partial}_{\mathcal{E}} =: \mathbb{E}^1$ $\quad$ $\bar{\partial}$ is holomorphic

$$\delta =: \mathbb{E}^0$$

$$\underline{\mathbb{E}^{k>1} = 0}$$

$$\mathbb{E}^{1\,2} = 0, \quad \mathbb{E}^{0\,2} = 0, \quad \mathbb{E}^0\mathbb{E}^1 + \mathbb{E}^1\mathbb{E}^0 = d\bar{\partial}_{\mathcal{E}} + \bar{\partial}_{\mathcal{E}} d = 0$$

<u>exercise</u> suppose $0 \to \mathcal{E}^0 \to \mathcal{E}^1 \to \mathcal{E}^2 \to 0$ SES of hol VB's

get cmplx of VB's

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{E}^0 \arrow[r] \arrow[d] & \mathcal{E}^1 \arrow[r] \arrow[d] & 0 \\ & 0 \arrow[r] & \mathcal{E}^2 \arrow[r] & 0 \end{tikzcd}$$

a quasi isomorphism of VB's. In $Ho\,\mathcal{P}_A$ these are isomorphic so construct the map backwards.

<u>example</u>: suppose $A \in \mathrm{Coh}\,X$. On cplx mnfld don't always have global resolutions by locally free sheaves. Take

$A \otimes_{\mathcal{O}} \underline{C}^\infty$ ← sheaf of C^∞-fnctns, will get resolution:

$$\begin{tikzcd} \cdots \arrow[r, "\mathbb{E}^0"] & E^1 \arrow[r, "\mathbb{E}^0"] \arrow[d, "\mathbb{E}^0"] \arrow[dl, "\mathbb{E}^1"'] & E^0 \arrow[r] \arrow[d, "\mathbb{E}^0"] \arrow[dl, "\mathbb{E}^1"'] & B \otimes_{\mathcal{O}} \underline{C}^\infty \arrow[d, "\bar{\partial}"] \\ \cdots \arrow[r] & E^1 \otimes_{A^0} A^1 \arrow[r] \arrow[d] & E^0 \otimes_{A^0} A^1 \arrow[r] \arrow[d] & B \otimes_{\mathcal{O}} A^1 \arrow[d] \\ \vdots & \cdots & \cdots & \vdots \end{tikzcd}$$

Thrm $B \in A^{0,2}X$, $\bar\partial B = 0$ B defines a class in $H^2(X,\mathcal{O}) \to H^2(X,\mathcal{O})$

B defines a topologically trivial Gerbe. $\downarrow$

$H^3(X,\mathbb{Z})$,

Let $A = (A^{0,\bullet}X, \bar\partial, B)$

$HoP_A \cong$ Derived Cat. of twisted coherent cohomology sheaves of weight 1.

Results related to this framework.

NC-Mukai Duality Take a torus $X = V/\Lambda$, V cplx vec. spc.

$B \in \Lambda^2 V^*$, $B = B^{2,0} + B^{1,1} + B^{0,2}$, B closed 2-form $B \in \Lambda^2 V^*$

$A_B = (A^{0,\bullet}X, \bar\partial, B) \leftarrow$ "twisting deformation of X"

$X^\vee = \bar V^\vee/\Lambda^\vee$ $C^\infty(X^\vee) \cong \mathcal{S}^*\Lambda$ define DGA $B^\bullet = \mathcal{S}^\bullet\Lambda \otimes \Lambda^\bullet V_{1,0}$

w/ $V_{1,0} =$ (1-eigenspace of $V \otimes \mathbb{C}$)

$\bar\partial\lambda = 2\pi i \lambda D'(\lambda)$ $D' : V \to V_{1,0}$ proj.

Lemma $\mathcal{B} \equiv (A^{0,\bullet}X^\vee, \bar\partial)$ Let $\sigma : \Lambda\times\Lambda \to U(1)$ be

$\sigma(\lambda_1,\lambda_2) = e^{2\pi i B(\lambda_1,\lambda_2)}$

form $\mathcal{B}_\sigma$: $[\lambda_1][\lambda_2] = \sigma(\lambda_1,\lambda_2)[\lambda_1+\lambda_2]$

the twisted algebra.

$\mathcal{B}_\sigma$ is highly non-commutative. $\bar\partial$ defined in same way.

Theorem : $\exists$ a deformed Poincaré object which implements an equivalence

$$HoP_{A_B} \xrightarrow{\sim} HoP_{\mathcal{B}_\sigma}$$

A deformed Fourier-Mukai transform.

Suppose B is non-degenerate. Then the support of objects in P_{A_B} must be isotropic w.r.t. B.

Think of B_σ as Ψdo's. These have to be isotropically supported and on the A_B-side these must be coisotropically supported } Gabber's Theorem