(Missed the previous class)

Morse Theory
7/2/08

Let R be a ring. Recall $\begin{matrix} K_1(R) \\ \| \\ GL_\infty(R)^{ab} \end{matrix}$

$= GL_\infty(R)/E(R)$.

Let (C_*, ∂) be a chain cmplx of left R-modules. Call (C_*, ∂) finite based free if specified a particular R-basis.

Suppose (C_*, ∂) acyclic (all homologies $= 0$ even at $* = 0$)

Since C_* consists of free R-modules, $\exists$ a contracting homotopy $h: C_* \to C_{*+1}$

$\partial h + \partial h = 1$.

Define $\tau(C_*) \in \tilde{K}_1(R) \equiv \text{Cok}[K_1(\mathbb{Z}) \to K_1(R)]$

$\tau(C_*)$: $\partial + h : C_{odd} \to C_{ev}$

w.r.t. basis, $\partial + h$ is represented by a matrix $A \in M_n R$

Lemma:

1) $A \in GL_n R$, 2) $\underset{\underset{\tau(C_*)}{\parallel}}{[A]} \in \tilde{K}_1(R)$ is indep. of h.

proof

Let k be another contracting homotopy.

Define $B =$ matrix representing $\partial + k : C_{ev} \to C_{odd}$

Set $\mu_n = (h_{n+1} - k_{n+1}) \circ k_n$, $\nu_n = (k_{n+1} - h_{n+1}) \circ h_n$

check: $(1+\mu_*)_{odd}$, $(1+\nu_*)_{ev}$,

$(\partial + h)_{odd} \circ (1+\mu_*)_{odd} \circ (\partial + k)_{ev}$,

$(\partial + k)_{ev} \circ (1+\nu_*)_{ev} \circ (\partial + k)_{odd}$,

All these are represented by lower-triangular matrices w/ 1's on the diagonal.

(hence represents 0 in K_1)

so,

$$[A] + [1+\mu_*]_{odd} + [B] = 0$$

$$\underbrace{[B] + [1+\nu_*]_{ev} + [A]}= 0$$

$$\Rightarrow [A] = -[B] \quad \text{in } \tilde{K}_1(R)$$

so, $[B], [A]$ indep of h, k.

If $f : (C_*, \partial) \to (D_*, \partial)$ is a homotopy of finite based free R-chain complexes,

$\tau(f) = \tau(\mathrm{Cone}(f))$ $\left[\begin{array}{c} \mathrm{Cone}(f)_k \equiv D_k \oplus C_{k-1} \\ \partial = \begin{pmatrix} \partial_D & f \\ 0 & -\partial_D \end{pmatrix} \end{array} \right]$

$0 \to D \to \mathrm{Cone}(f) \to C[-1] \to 0$

exact seq. of chain complexes

$\Rightarrow \exists$ LES (snake lemma)

$$\to H_k(D) \to H_k(\mathrm{Cone}(f)) \to H_{k-1}(C_*) \xrightarrow{f_*} H_{k-1}(D_*) \to$$

<u>Definition</u> (Reidemeister Torsion)

X be a finite CW-complex and let $\pi = \pi_1(X)$.

Let V be an orthogonal rep'n of π. (fin. dim'l.)

Let $C_*(X; V) \equiv C_*(\tilde{X}, R) \otimes_{R\pi} V$

Assume X is V-acyclic, i.e. $H_*(X, V) \equiv H_*(C_*(X;V)) = 0$ $\forall *$

i.e. $C_*(X;V)$ is a finite, based free, contractible R-module.

Define ρ the Reidemeister torsion

$$\rho(X;V) = \tau(C_*(X;V)) \in K_1(\mathbb{R}) \overset{\det^2}{\cong} \mathbb{R}_{>0}$$

Since V is an orthogonal representation, if I change basis elt. of $C_*(\tilde{X})$ by g, multiply ρ by $\det^2(g)=1$, so ρ is indep. of the lifts.

Classification of lens spaces:

Let E be a Hermitian vec. spc π acting by unitaries. Take $S(E)$ $(\cong S^{2n-1})$. π acts on $S(E)$. Let $\pi = \mathbb{Z}/\ell\mathbb{Z}$. The rep π is called free if its action on $S(E)$ is free. (So assume π is free) define $L(E) = S(E)/\pi$

$S(E) \to L(E)$ is a covering.

$\pi_1(L(E)) = \pi$

$\pi_k(L(E)) = \pi_k(S(E)) \quad k>1.$

The irreducible reps. of π are all of the form:

$g \longmapsto \left(e^{\frac{2\pi i k}{\ell}}\right)$. $\qquad E = V_{k_1} \oplus \cdots\cdots \oplus V_{k_n}$

$L(\ell, k_1, \cdots, k_n)$ are all lens. spaces. (the k_i's must be s.t. $(k_i, \ell) = 1$)

<u>The homotopy classification of Lens spaces:</u>

$$L(\ell; k_1, \cdots, k_n) \simeq L(\ell'; k_1', \cdots, k_n') \Longleftrightarrow$$

$$\ell = \ell' \text{ and } \underbrace{\prod k_i}_{\in (\mathbb{Z}/\ell\mathbb{Z})^*} = \pm e^n \prod k_i', \quad e \in (\mathbb{Z}/\ell\mathbb{Z})^*$$

<u>Lemma</u> 1) E free, unitary π-module.
V a orthogonal repn of π s.t.
$V^\pi = \{0\}$
$\shortparallel$
$\{v \in V \mid gv = v \ \forall g\}$ Then $C_*(L(E); V)$ is acyclic.
(so Reidemeister Torsion defined)

2)

$$\rho(L(E_1 \oplus E_2); V) = \rho(L(E_1); V)\, \rho(L(E_2); V)$$

$$\rho(L(E); V_1 \oplus V_2) = \rho(L(E); V_1)\, \rho(L(E); V_2)$$

pf of D: claim: If X is any finite CW cmplx, s.t. $\pi_1 X = \pi$ and π acts trivially on $H_p(\tilde{X})$ $\forall p$ and V is an orthogonal rep'n $V^{\pi} = 0$

$C_*(X; V)$ is acyclic

$\shortparallel$

$C_*(\tilde{X}; \mathbb{R}) \otimes_{\mathbb{R}\pi} V$

because $C_*(\tilde{X}; \mathbb{R})$ is free (no ext terms)

$$H_*(C_*(\tilde{X}; \mathbb{R}) \otimes_{\mathbb{R}\pi} V) \cong \underbrace{H_*(C_*(\tilde{X}; \mathbb{R}))}_{\text{trivial } \pi\text{-module.}} \otimes_{\mathbb{R}\pi} V$$

$$\text{RHS} = (\mathbb{R} \oplus \mathbb{R} \oplus \cdots \oplus \mathbb{R}) \otimes_{\mathbb{R}\pi} (W_1 \oplus W_2 \oplus \cdots \oplus W_f)$$

irreps.

$$= \bigoplus \underbrace{\mathbb{R} \otimes_{\mathbb{R}\pi} W_i}$$

$\{0\}$ since W_i irrep. and $\mathbb{R}$ trivial π-module.

$$= \{0\}.$$