Morse Theory Notes 7/7/08

$V_1, V_2$ two Hermitian vec. spaces w/ free unitary $\pi = \mathbb{Z}/t\mathbb{Z}$

$L(V_1) = S(V_1)/\pi$ (free means free on the sphere)

$V_1 \cong V_{k_1} \oplus \cdots \oplus V_{k_n}$,

$V_{k_i} \cong \mathbb{C}$ the one-dim'l rep'n. $g \in \pi$ (generator) acts by $e^{2\pi i k_i/t}$ $(k_i, t) = 1$ (to be free)

$V_2 \cong V_{l_1} \oplus \cdots \oplus V_{l_n}$ $\dim_{\mathbb{R}} L(V_i) = 2n-1$.

$\longrightarrow$ classification! (homotopy)

$L(V_1) \simeq L(V_2) \iff \exists\, e \in (\mathbb{Z}/t\mathbb{Z})^\times$ s.t.

$$\prod_{i=1}^{n} k_i \equiv e^n \prod_{i=1}^{n} l_i \mod t$$

<u>Theorem</u> TFAE $(2n-1) \geq 3$

1) $\exists$ an automorphism $\alpha: \pi \to \pi$ s.t. $V_1 \cong \alpha^* V_2$ as orthogonal rep'ns
2) $\exists$ diffeo $L(V_1) \to L(V_2)$
3) $\exists$ homeo $L(V_1) \to L(V_2)$
4) $\exists$ an auto $\alpha: \pi \to \pi$ s.t. for any orthogonal rep $W$ w/ $W^\pi = 0$, $\rho(L(V_1); W) = \rho(L(V_2), \alpha^* W)$

5) $\exists\ \alpha : \pi \to \pi$ s.t. for any 1-dim'l unitary rep. of $W$

$$\rho(L(V_1); \operatorname{res} W) = \rho(L(V_2), \operatorname{res} \alpha^* W)$$

Pf  1) $\Rightarrow$ 2) $\Rightarrow$ 3) $\Rightarrow$ 4) $\Rightarrow$ 5) are obvious.

5) $\Rightarrow$ 1): $\alpha(g) \in (\mathbb{Z}/t\mathbb{Z})^*$ ($g \in \mathbb{Z}/t\mathbb{Z}$ a gen.)

$\alpha(g) = f$, w/ $(f, t) = 1$  $\alpha(g) = g^f$

$S(V_k)$ some particular $k_i$ as above

$C_*(S(V_k); \alpha^* W)$ $\mathbb{Z}\pi$-chain cmplx.

$C_0, C_1$ are free $\mathbb{Z}\pi$ modules of rk 1.

$\partial : C_1(S(V_k); \alpha^* W) \longrightarrow C_0(S(V_k); \alpha^* W)$

is just mult. by $g^f - 1$.

Now suppose that $g$ acts on $W$ by mult. by $\zeta \neq 1$ a $t^{th}$ root of unity.

$$\rho(L(V_k); W) = \det{}^2(g^k - 1)$$

$$= \|\zeta^k - 1\| = (\zeta^k - 1)(\zeta^{-k} - 1)$$

For $L(V_1 \cong V_{k_1} \oplus \cdots \oplus V_{k_n})$

so, $\rho(L(V_1); W) = \prod_{i=1}^{n} (\zeta^{k_i}-1)(\zeta^{-k_i}-1)$

$$\| $$

$\rho(L(V_2); \alpha^* W) = \prod_{i=1}^{n} (\zeta^{fl_i}-1)(\zeta^{-fl_i}-1)$

$\Rightarrow$ (Franz' Independence Lemma)

Let $t > 2 \in \mathbb{Z}$, $S =$ set of $\{ j \in \mathbb{Z} \mid 0 < j < t, (j,t) = 1 \}$

$= (\mathbb{Z}/t\mathbb{Z})^*$

Let $a_j \in \mathbb{Z}$, $j \in S$

s.t. $\sum a_j = 0$, $a_j = a_{t-j}$ for $j \in S$ and

$$\prod (\zeta^j - 1)^{a_j} = 1 \quad \forall \text{ roots of } 1, \zeta \neq 1$$

then, $a_j = 0 \ \forall j$.

<u>exercise</u> show the independence lemma implies the following:

$\exists$ permutation $\sigma \in \Sigma_n$, signs $\varepsilon_i = \{\pm 1\}$ s.t.

$k_i = \varepsilon_i f l_{\sigma(i)} \mod t$.

This implies that $V_1 \cong \alpha^* V_2$.

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Corollary

$$L(t;k_1;\cdots;k_n) \overset{\text{diffeo}}{\cong} L(t;l_1,\cdots,l_n)$$

$\Longleftrightarrow$ $\exists f \in (\mathbb{Z}/t\mathbb{Z})^*$, $\varepsilon_i = \{\pm 1\}$, perm. $\sigma \in \Sigma_n$,

s.t. $k_i = \varepsilon_i f l_{\sigma(i)} \mod t$.

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examples:

From homotopy classification:

1) $L(5;1,1)$, $L(5,2,1)$ have the same homotopy and homology but aren't homotopy equivalent.

2) $L(7;1,1)$, $L(7;2,1)$ are homotopy equiv but not homeomorphic.

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Back to Morse Theory:

Let $X$ be a diff'ble mnfld, $f$ a function.

Let $g$ be a metric on $X$,

$\nabla f$ VF: $g(\nabla f, Y) = df(Y)$.

Let $\psi_s$ = 1-param. flow for $-\nabla f$.

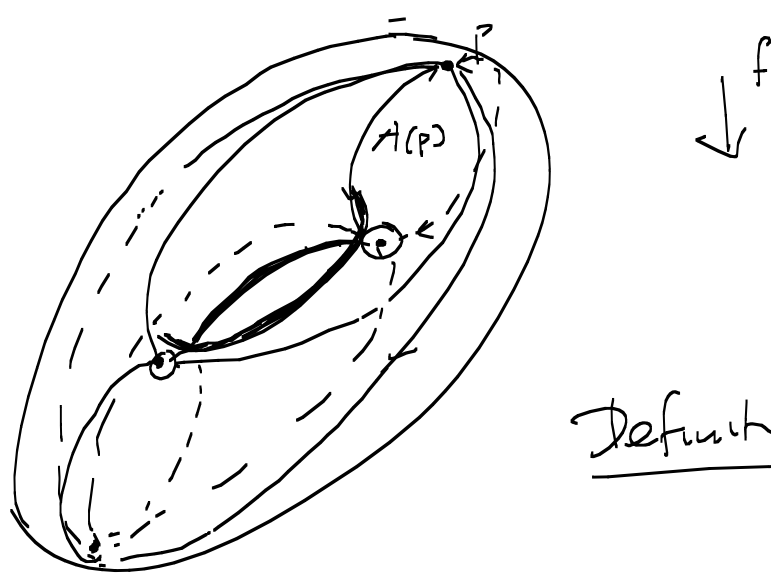
($\psi_s : X \to X$ difeo. (X complete so $s \in (-\infty, \infty)$))

$\left( \frac{d\psi_s}{ds} = -\nabla f. \right)$

If $p$ is a crit. pt. of $f$, define two spaces:

<u>Define</u> $\mathcal{D}(p) = \{x \in X \mid \lim_{s \to -\infty} \psi_s(x) = p\}$ = descending mnfld.

$\mathcal{A}(p) = \{x \in X \mid \lim_{s \to \infty}$ " $\}$ = ascending mnfld.




<u>Definition</u>

The pair $(f, g)$ are called Morse-Smale if any 2 crit. pts. $p \neq q$ $\mathcal{D}(p) \pitchfork \mathcal{A}(q)$.

If $p, q$ are crit. pts, a flow line from $p$ to $q$ is a $\gamma: \mathbb{R} \to X$ s.t. $\lim_{s \to -\infty} \gamma(s) = p$

$\gamma'(s) = -\nabla f(\gamma(s))$ $\lim_{s \to \infty} \gamma(s) = q$

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Let $\tilde{\mathcal{M}}(p,q)$ = the flow lines from $p$ to $q$.

$\mathbb{R}$ acts on $\tilde{\mathcal{M}}(p,q)$ by $t \cdot \gamma(s) = \gamma(s+t)$

Define $\mathcal{M}(p,q) = \tilde{\mathcal{M}}(p,q)/\mathbb{R}$

$\mathcal{M}(p,q) = \mathcal{D}(p) \cap \mathcal{A}(q)/\mathbb{R}$, where $\mathbb{R}$ acts by $\psi_s$.

$\dim \mathcal{M}(p,q) = \operatorname{ind}(p) - \operatorname{ind}(q) - 1$.

(Schwarz Morse Homology)