Morse Thy
7/8/18 Block

X closed mnfld.

$f: X \to \mathbb{R}$ Riemannian metric

$-\nabla f$, ψ_s the flow of $-\nabla f$

If $p \in Crit(f)$ $\mathcal{D}_p(f) = \{x \in X \mid \lim_{t \to -\infty} \psi_s = p\}$

$\mathcal{A}_p(f) = \{ \quad " \quad \mid t \to \infty \}$

If p is non-degenerate, $\mathcal{D}(p) \cong D^{\operatorname{ind} p}$

$\mathcal{A}(p) \cong D^{\dim X - \operatorname{Ind} p}$

(f, g) is called Morse-Smale if for any pair $p, q \in Crit(f)$

$\mathcal{D}(p) \pitchfork \mathcal{A}(q)$

<u>Statement</u> for generic pairs (f, g) is Morse-Smale

$\tilde{\mathcal{M}}(p, q) =$ flow lines from p to q $\equiv \{\gamma \mid \lim_{s \to \infty} \gamma(s) = p$

$\lim_{s \to -\infty} \gamma(s) = q$

$\gamma'(s) = -\nabla f$

$\tilde{\mathcal{M}}(p, q) \equiv D(p) \cap \mathcal{A}(q)$

$\dim \tilde{\mathcal{M}}(p, q) = \operatorname{ind} p - \operatorname{ind} q$

$\mathcal{M}(p,q) = \tilde{\mathcal{M}}(p,q)/\mathbb{R}$ obvious action

$\mathbb{R}$ acts on $\tilde{\mathcal{M}}(p,q)$ $\quad (t\cdot\gamma)(s) = \gamma(s+t)$

$\dim \mathcal{M}(p,q) = \operatorname{ind} p - \operatorname{ind} q - 1$

let's orient $\mathcal{M}(p,q)$. Pick an orientation of each $\mathcal{D}(p)$. Let $\gamma \in \mathcal{M}(p,q)$, $T\mathcal{D}(p) \cong T(\mathcal{D}(p) \cap \mathcal{A}(q)) \oplus TX/\mathcal{A}(q)$

by transversality

$$\cong T_\gamma(\tilde{\mathcal{M}}(p,q)) \oplus TX/\mathcal{A}(q) \cong \underbrace{T_\gamma(\mathcal{M}(p,q)) \oplus T\gamma \oplus TX/\mathcal{A}(q)}_{\cong\, T\mathcal{D}(q)}$$

Put an orientation on $T_\gamma(\mathcal{M}(p,q))$ making these isomorphisms orientation-preserving

<u>Definition</u>: a <u>mnfld</u> <u>w/</u> <u>corners</u> is a "manifold" modelled on $\mathbb{R}^{n-k} \times [0,\infty)^k$

<u>Theorem</u> X, (f,g) Morse-smale, then for any two crit pts p,q, $\mathcal{M}(p,q)$ has a compactification on a mnfld w/ corners.

where k^{th} stratum, $\mathcal{M}_k(p,q)$ has the props:

$$\mathcal{M}_0(p,q) = \mathcal{M}(p,q), \quad \mathcal{M}_1(p,q) = \bigcup_{\substack{r\in Crit(f)\\ q\neq r\neq p}} \mathcal{M}(p,r)\times\mathcal{M}(r,q)$$

$$\mathcal{M}_\ell(p,q) = \bigcup_{\substack{r_1\cdots r_\ell \in Crit(f)\\ q\neq r_i\neq p\neq r_j}} \mathcal{M}(p,r_1)\times\mathcal{M}(r_1,r_2)\times\cdots\times\mathcal{M}(r_\ell,q)$$

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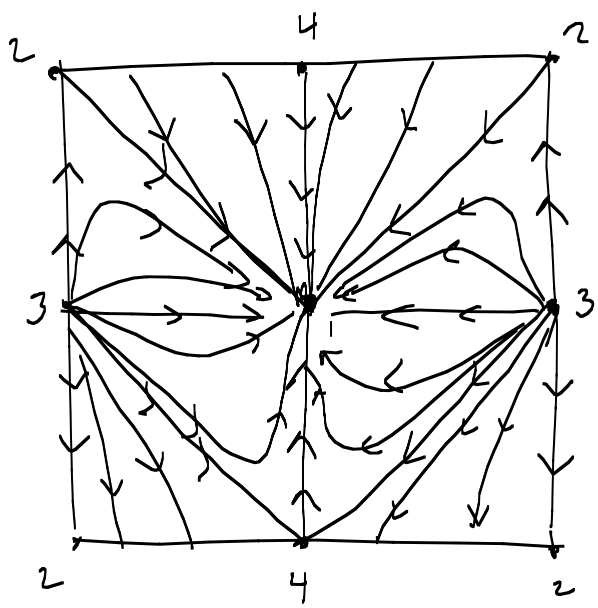
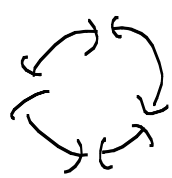
example! Torus



$\mathcal{M}(3,4) =$ two pts.

$\mathcal{M}(3,2) =$ two pts.

$\mathcal{M}(3,1) =$



$$\mathcal{M}(3,1)_1 = \mathcal{M}(3,2)\times\mathcal{M}(2,1) \cup (\mathcal{M}(3,4)\times\mathcal{M}(4,1))$$

$\mathcal{M}(2,1) =$ two pts.

$\mathcal{M}(4,1) =$ two pts.

Steps to proof:

1) given any sequence of flow lines, $\exists$ a sequence which converges to a union of flow lines.

2) If you have a union of flow lines, such that if you perturb, you get a genuine flow line.

If ind. $p =$ ind $q + 1$, then $\dim \mathcal{M}(p,q) = 0$, and cpct.

therefore finite

Morse Complex $\quad C_k^M(f,g) \equiv \mathbb{Z}\, \mathrm{Crit}_k(f)$

$$\partial^M : C_k^M(f,g) \to C_{k-1}^M(f,g)$$

$$\partial^M(p) = \sum_{q \in \mathrm{Crit}_{k-1}(f)} \#(\mathcal{M}(p,q))\, q$$

lemma $\partial^M \circ \partial^M = 0$ $\quad$ pf $p \in \mathrm{Crit}_k(f)$ $\quad \partial^M \circ \partial^M(p) = \sum_{q \in \mathrm{Crit}_{k-2}} ?\, q$

$\overline{\mathcal{M}(p,q)}$ ind $p =$ ind $q + 2$

$\overline{\mathcal{M}(p,q)}_0 = \mathcal{M}(p,q)$, $\quad \overline{\mathcal{M}(p,q)}_1 = \bigcup_r \mathcal{M}(p,r) \times \mathcal{M}(r,q)$

$\overline{\mathcal{M}(p,q)_k} = \phi$. $k > 1$ i.e. $\overline{\mathcal{M}(p,q)}$ is a mnfld

w/ ∂, $\dim \mathcal{M}(p,q) = \operatorname{ind} p - \operatorname{ind} q - 1 = 1$.

i.e. $\overline{\mathcal{M}(p,q)}$ is a 1-mnfld w/ ∂.

<u>Definition</u> The <u>Morse Homology</u> is the homology of this complex.

<u>Fact</u>: $H^M_*(f,g) \cong H_*(X)$ if (f,g) is MS.