Teleman ext TQFTs | 10/28/08

More on weak ∞-categories:

Several good models:

- Quasi Cats (Joyal)
- Segal Cats. (Hirschowitz + Simpson)
- complete Segal spc. (Rezk)

Model example: If X top. spc. Then $\Pi_{\leq n} X$ should be an n-groupoid ($=(n,0)$-cat)

prelim Def: An $(\infty,0)$-cat is a top. spc. (or SSet or Kan Cmplx)

Note: In fact, we want $((\infty,0)$-cats$/$equivs$) =$

$= ($top spcs $/$ homotopy equivs$)$

But the actual $(\infty,0)$ cats will be e.g. fibrant top. spaces of SSets ($=$Kan cmplxes)

Next: an $(\infty,1)$-cat. is a topological cat.

- good defn. but hard to work with.

Different models: (come from Thus classifying spc. constr.)

Recall: If $\mathcal{C}$ small cat. $N\mathcal{C}$ nerve of $\mathcal{C}$

$$\left(\quad N\mathcal{C}: \quad \Delta^{op} \to (Set) \qquad \text{ob} \right.$$
$$[n] \to Fun([n], \mathcal{C}) \qquad \text{Mor}$$

$N: (Cat) \to (SSet)$ is a functor

<u>Theorem</u>: N is fully faithful and the essential image consists of all SSets X s.t. $\forall k$ the Segal map $X_k \to X_1 \times_{X_0} \cdots \times_{X_0} X_1$ is an isom.

Tamsamani n-categories

This is a notion of an n-cat. built by induction:

(i) base step: 0-cats. are sets $\Rightarrow$ $(0cat) \equiv (Set)$

$(Set) \xrightarrow[\tau_0 = id]{\delta = id} (0\text{-}Cat)$ = fully faithful (discrete cats)

$(0\text{-}cat) \to (Set)$ - truncation.

(ii) suppose we have $((n-1)\text{-}Cat)$ = Cat. of $(n-1)$-cats. and

$\delta: (Set) \to (n-1)(cat)$ fully faithful

$\tau_0: (n-1)(cat) \to (Set)$

a class $\{(n-1)\text{-equivalences}\}$ of morphisms, and products + coproducts, finite limits + colimits.

s.t.
- $\tau_0 \circ \delta = id_{(Set)}$
- τ_0: $(n-1)$-equivs $\mapsto$ bijections
- τ_0 preserves finite products

- If A, B $(n-1)$-cats, B-discrete, $f: A \to B$ - morphism of $(n-1)$-cats

$\Rightarrow$

Construction

- $(n\mathrm{Cat})$ = cat. of all $X : \Delta^{op} \to ((n-1)\mathrm{Cat})$

s.t.
- X_0 is discrete
- $\forall k \geq 1$ the Segal map

$$X_k \to X_1 \times_{X_0} \cdots \times_{X_0} X_1$$

is an $(n-1)$-equivalence.

Then the natural map

$$A \to \coprod_{x \in B} A_x$$

is an isom. of $(n+1)$-categories

- $\delta_n : (\mathrm{Set}) \to (n\mathrm{Cat})$

$S \to \delta_n(S)$ is the constant, discrete $(n-1)\mathrm{Cat}$ $\delta_{n-1}(S)$

$\delta_k(S)_k = \delta_{n-1}(S)$ + identities

we will define $\tau_1 : (n\mathrm{Cat}) \to (\mathrm{Cat})$ and

$\tau_0 : (n\mathrm{Cat}) \to (\mathrm{Set})$ $\qquad \tau_{0_{n\mathrm{Cat}}} = \tau_{0_{\mathrm{Cat}}} \circ \tau_1$

$\tau_1 : (n\mathrm{Cat}) \to (\mathrm{Cat})$

$$[A : \Delta^{op} \to (n-1)\mathrm{Cat}] \to [\tau_0 \circ A : \Delta^{op} \to (\mathrm{Set})]$$

- equivalences: note: $\forall k \geq 1$, the map

$$A_k \to \underbrace{A_0 \times \cdots \times A_0}_{k+1 \text{ times}}$$

induced from the $(k+1)$-maps $[0] \to [k]$ is a map of $(n-1)$-cats.

we get an iso $A_k \cong \coprod_{\underline{a} \in A_0^{\times k}} A_{(a_0, \dots, a_k)}$

So if $f: A \to B$ is a map of $(n-1)$-cats $\Longrightarrow$ f is an n-equivalence

iff
- $\forall\, x, y \in A_0$ $\quad A_{(x,y)} \to B_{(f(x), f(y))}$ is an equiv. of $(n-1)$ cats.

- $\tau_1(f): \tau_1 A \to \tau_1 B$ is an equivalence

Can fudge the definition to say what an n-groupoid is:

$((0,0)\text{ cat}) := 0\text{-Cat}$

if $(n-1,0)$-cat is defined and is a full subcategory in $(n-1)$Cat.

then define $((n,0)\text{-Cat}) =$ all n-cat. A s.t.

- $\tau_1 A =$ groupoid
- $\forall\, x, y \in A_0$ $\quad A(x,y)$ is $(n-1)$ groupoid.

The homotopy cat. of $((n,0)\text{ Cat})$, $\mathrm{Ho}((n,0)\text{Cat}) =$ localization of $(n,0)$-cat by n-equivalences

<u>theorem</u> (Tamsamani)

There are natural functors $(n,0)\text{Cat} \underset{\pi_{\leq n}}{\overset{B}{\rightleftarrows}} (\text{Top})$

which induce a natural equivalence

$$\mathrm{Ho}\,((n,0)\text{Cat}) \cong \mathrm{Ho}((\text{Top})_{\leq n})$$