Porter ExtTFTs 11/5/8

Hirschowitz - Simpson. Segal categories

Idea: If you have a good notion of an ∞-category, then if A is an (∞, n)-category for that notion, then the cat. of n-morphisms of A are all $(\infty, 0)$-cats. We have good models for these, so we can use these models. We want to put together simplicial sets and Tamsamani's defn.

Define define a cat. of n-Segal categories : $(n\mathrm{SeCat})$
+ equivalences of n-segal cats.
+ $\delta : (\mathrm{Set}) \longrightarrow (n\mathrm{SeCat})$
$\tau_0 : (n\mathrm{SeCat}) \longrightarrow (\mathrm{Set})$

inductively in n

- $(0\,\mathrm{SeCat}) = (\mathrm{SSet})$
  $\delta : (\mathrm{Set}) \to (\mathrm{SSet})$ - constant SSets
  $\tau_0 : (\mathrm{SSet}) \to (\mathrm{Set})$
  $\tau_0 = \pi_0$
  0-equivalences $\to$ weak equivs of SSets.

Suppose we have defined the cat $(n-1\,\mathrm{SeCat})$, $(n-1)$-equivs, δ, τ_0

Define $(n\,\mathrm{SeCat})$ = Category of all functors

$A : \Delta^{op} \to (n\text{-}1\,SeCat)$ s.t. A_0 discrete, all Segal maps

$A_k \to A_1 \times_{A_0} A_1 \times \cdots \times_{A_0} A_1$ are $(n\text{-}1)$ equivalences

Properties:

(1) $(n\,Cat) \subset (n\,SeCat)$ as a full subcategory

$Ho(n\,Cat) \subset Ho(n\,SeCat)$ are all full subcats.

(2) look at $(1SeCat)$ These are functors

$A : \Delta^{op} \to (SSet)$ (bisimplicial category)

s.t. the Segal maps are weak homotopy equivalences of SSets.

Note: $Ho(1Cat) \subset Ho(1SeCat)$

$\uparrow$ the full subcat. of $A \in (1SeCat)$, $\pi_i(A) = 0$, $i > 0$

Theorem (Dwyer-Kan-Smith)

simplicial cats.

(1) The functor $(S\,Cat) \longrightarrow (1SeCat)$ is fully faithful

and the essential image consists of all 1SeCats s.t. the Segal maps are isomorphisms.

(2) $Ho(SCat) \rightsquigarrow Ho(1SeCat)$ is an equivalence

(3) Model structure on Segal Cats:

Problem: (nSeCat) cannot be a model category. (Doesn't have limits or colimits.) Need to enlarge it to a closed model cat. which has the same homotopy category.

Defn The cat. of (nSe preCat) inductively:

for n=0, set (0 Se preCat) = (SSet) = (0SeCat)

if ((n-1)SepreCat) defined, take (n Se preCat) to be the cat. in simplicial objects in (n-1 Se preCat) but don't impose the condition Segal maps are equivs. Just require A_0 discrete. .... etc.

Theorem (Pellisier-Simpson)

∃! closed Model structure w/ given weak equivs. s.t.

(1) The fibrant objects are (nSeCats) ⊂ (nSepreCats)

(2) The cofibrations are the monomorphisms

(3) weak equivs between fibrant objects are n-Segal equivalences

(4) ((n-Segal preCats), ×, 𝟙) is a monoidal closed model category.

$\Longrightarrow \quad \underset{\substack{\uparrow \\ \text{fibrant}}}{\bar{X}} \cdot (\cdot) : (n\mathrm{SeCat}) \circlearrowleft$ preserves weak equivalences.

Remark: (1) If C - closed model cat. then it is a monoidal closed model cat

If we have a monoidal structure

$(C, \Box, \mathbb{1})$ s.t.

$\Box : C \times C \longrightarrow C$ is a Quillen bifunctor

If we take the canonical morphism $0 \to \mathbb{1}$ and we factor

$$\underset{\substack{\uparrow \\ \text{trivial fibration}}}{0 \longrightarrow} Q\mathbb{1} \underset{\substack{\uparrow \\ \text{cofibration}}}{\xrightarrow{r}} \mathbb{1}$$

then for any cofibrant object $x \in C$

$$\Longrightarrow \left\{ \begin{array}{l} X \Box Q\mathbb{1} \xrightarrow{1 \Box r} X \\ Q\mathbb{1} \Box X \xrightarrow{r \Box \mathbb{1}} X \end{array} \right\}$$ are weak equivalences.

(2) not all (n-Segal Cats) are fibrant.

(3) weak equivalences between general (n Segal preCats) are hard to compute.

(4) (nSe preCat) is homotopy cocomplete, i.e. if I small cat

$$Ho((n\mathrm{SePreCat}))^I \longrightarrow Ho(\mathrm{SeCat})$$

const. diagram functor

has a left adjoint.

(5) (n Se PreCat) has derived internal homs

$\forall a,b \quad \mathbb{R}\mathrm{Hom}(a,b) \xrightarrow{\sim} \mathcal{H}om(a, Rb)$

Rb = fibrant replacement of b.

⟹ all fibrant n Segal Cats + internal homs give an (n+1 Se Cat)

(6) Can define 1 Segal localizations of a Cat C along a smooth Cat $W \subseteq C$ subcategory.