Erik Van Erp Index Thy + Elliptic Operators I 9/16/18

M = closed, smooth mnfld, P = diff'l operator, elliptic

choose coords: $\mathcal{U} \subseteq M$
$\mathcal{U} \to \mathbb{R}^n$

locally: $P = \sum_{|\alpha| \leq d} a_\alpha \partial^\alpha$ α is multi-index $(\alpha_1 \cdots \alpha_n)$ non-neg. integers. $|\alpha| \equiv \sum \alpha_i$

$$\partial^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1}\left(\frac{\partial}{\partial x_2}\right)^{\alpha_2} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}$$

$a_\alpha(x)$ = smooth func. of x, $d \equiv$ order of P

For bundles, $\{E, F\}$

$$P : C^\infty(E) \longrightarrow C^\infty(F)$$

after trivializing over a patch, a_α's become matrix-valued

Symbols: Fourier Transform: $\frac{\partial}{\partial x_j} \longmapsto i\xi_j$ ← coordinate on $\hat{\mathbb{R}}^n \cong (\mathbb{R}^n)^*$

$\sigma_{full}(P)$ (in these coords) =

$$= \sum_{|\alpha| \leq d} a_\alpha(x)(i\xi)^\alpha$$

$(x, \xi) \in \mathbb{R}^n \times \hat{\mathbb{R}}^n$

As stated, this is not invariant at all. But the principal part does:

highest order part $\boxed{\sigma(P) \equiv \sum_{|\alpha|=d} a_\alpha(x)(i\xi)^\alpha}$

→ This is smooth in x, homogeneous poly in ξ of order d

→ Invariant as a function on T^*M with values in the bundle $\mathrm{End}(\pi^*E, \pi^*F)$. ($\pi : T^*M \to M$.)

P is elliptic if its symbol is invertible.

(as a func on co-sphere bundle since it is homogeneous on T^*M and $=0$ at 0)

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examples: • $d+d^*$: deRham operator.

• Δ: any of the various Laplacians

• $\bar\partial + \bar\partial^*$: Dolbeault operator.

Symbols obey a calculus:

P, Q order d, d' then $\sigma(PQ) = \sigma(P)\sigma(Q)$

Key theorem !!: P is elliptic, M cpct. ($\partial M = \emptyset$)

Then, consider $D \equiv \begin{pmatrix} 0 & \bar{P} \\ \bar{P}^* & 0 \end{pmatrix}$ on $E \oplus F$

i) D is self-adjoint on $L^2(E \oplus F)$

(note D is unbounded so being self-adjoint is a difficult notion)

(and $\bar{P}$ is the "closure" of P to L^2 this can always be done for a diff'l op P)

($P^* \equiv$ formal adjoint of P)

Theorem: $\overline{P^*} = \bar{P}^*$ the Hilbert spc. adjoint. (hard)

(Import of self-adjointness: $\operatorname{Spec} D \subseteq \mathbb{R}$), (note D is also elliptic)

ii) D has a discrete spectrum, no accumulation pts.

iii) Eigenspaces are all fin. diml., and all eigenfunctions are C^∞.

In particular, D is Fredholm. (and also P)

$D: L^2(E \oplus F) \circlearrowleft$ unbdd.

For bdd op. $T: H \circlearrowleft$ it is Fredholm if T has a parametrix:

i.e. $\exists\, Q$ s.t. $TQ = 1 + R_1$ w/ R_i compact operator.
$QT = 1 + R_2$

$$0 \to \underset{\substack{\text{compact}\\ \text{operators}}}{K(\mathcal{H})} \to B(\mathcal{H}) \longrightarrow \underset{\substack{\text{Calkin}\\ \text{algebra}}}{Q} \longrightarrow 0$$

closed two-sided ideal

T Fredholm $\Leftrightarrow$ $T \longmapsto [T]$ invertible.

<u>Theorem</u> Topological group G of invertible elts. in Q has $G_e \equiv$ connected comp. of e

Then $0 \longrightarrow G_e \to G \longrightarrow \mathbb{Z} \to 0$

<u>Definition</u>:
The <u>Fredholm index</u> of $T \equiv \varphi([T]) \in \mathbb{Z}$.

<u>Proposition</u>: $\text{Index}(T) = \dim \operatorname{Ker} T - \underbrace{\dim \operatorname{Coker} T}_{= \dim \operatorname{Ker} T^*}$

<u>Comment</u>! what about P elliptic on $L^2(E \oplus F)$ unbdd?

Call and <u>unbdd.</u> op. <u>Fredholm</u> if
i) $\operatorname{Ker} P$ fin. dim.
ii) $\operatorname{Coker} P$ fin. dim.
iii) closed range.

So, P elliptic on cpct. closed mfld M $\Rightarrow$ P Fredholm

$\Rightarrow$ P has index, and furthermore,

(independence by cpct. perturbation) $\Rightarrow$ the index only depends on $\sigma(P)$. In fact only on homotopy class $[\sigma(P)]$.

Index problem: how does $\sigma(P)$ depend on index(P)?

$\sigma(P)$ is a section over S^*M in $\mathrm{Aut}(\pi^*E, \pi^*F)$.

sloppily: $S^*M \longrightarrow GL(2n, \mathbb{C})$ (locally)

We want to derive a topological formula for the Fredholm index as a function of $[\sigma(P)]$.

Solution: Atiyah-Singer Formula:

$$\mathrm{Index}\, P = \int_{T^*M} \mathrm{Ch}[\sigma(P)] \wedge \pi^* \mathrm{Td}(M)$$