Van Erp Subelliptic operators 9/25/08

M cpt. closed, $E, F \xrightarrow[\pi]{\text{VB's (equal rk)}} M$, $P: C^\infty(E) \to C^\infty(F)$

$\sigma(P) \in C^\infty(\mathrm{Hom}(\pi^*E, \pi^*F)$ over $T^*M)$

(this is 0 at 0 and scaling invariant)

If $\sigma(P)$ invertible for $\xi \neq 0$, then

Atiyah-Singer:

1) Index P depends only on homotopy type of $\sigma(P)$

2) $\mathrm{Index}\, P = \displaystyle\int_{T^*M} ch(\sigma(P)) \wedge Td(M)$

as a section $S^*M \to GL(k, \mathbb{C})$ locally not in $M_k\mathbb{C}$

(note $ch(\sigma(P)\sigma(Q)) = ch(\sigma(P)) + ch(\sigma(Q))$)

If P not elliptic, AS formula is meaningless.

fact:

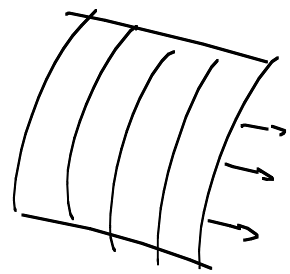
[hypoelliptic operators / subelliptic operators turn out to be Fredholm on compact M.]

(note: on M cpt.
hypoelliptic $P \Leftrightarrow$ u, v distribs.
$Pu = v$, if $v \in C^\infty$ in $U \overset{op}{\subseteq} M$
$\Rightarrow u \in C^\infty$ in U)

General principle: AS formula has wider applications.

<u>Hörmander ('68)</u>: (M, H) M cpct, oriented, foliated $\partial M = \emptyset$.

$H \subset TM$, $\underbrace{[H, H] \subseteq H}_{\text{integrable}}$



We can design operators which are elliptic on leaves and non-elliptic transversally.

(Think heat operator)

Pick coordinates $x = (x_1 \dots x_p, x_{p+1} \dots x_{p+q}) \in \mathbb{R}^p \times \mathbb{R}^q$

w/ $\frac{\partial}{\partial x_1} \cdots \frac{\partial}{\partial x_p} \in H$.

tangent to leaves.

<u>example</u>: $\left[\frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_p^2} + a_1 \frac{\partial}{\partial x_{p+1}} \cdots a_q \frac{\partial}{\partial x_{p+q}} + b =: P\right]$

$\sigma(P) = -\xi_1^2 - \cdots - \xi_p^2$

let $\frac{\partial}{\partial x_1} \dots \frac{\partial}{\partial x_p}$ have order 1, let $\frac{\partial}{\partial x_{p+1}} \cdots \frac{\partial}{\partial x_{p+q}}$ have order 2.

$\mathcal{P} \equiv$ alg. of diff'l ops. on M. This is filtered by order of the operators.

$$P^k \cdot P^\ell \subseteq P^{k+\ell} . \qquad \text{Let } P_\bullet = \text{assoc. graded } (P)$$

$$P_\bullet = \bigoplus_{j=0}^{\infty} P_{j+1} / P_j$$

In this modification, $\sigma_H(P)(x,\xi) \equiv \sum_{\|\alpha\|=d} a_\alpha(x)(i\xi)^\alpha$

$P = \sum a_\alpha \partial^\alpha$

w/ $\|\alpha\| = \alpha_1 + \dots + \alpha_p$

$+ 2(\alpha_{p+1} + \dots + \alpha_{p+\xi})$

so,

$$\sigma_H(P) = -\xi_1^2 - \dots - \xi_p^2 + a_1 i \xi_{p+1} + \dots + a_\xi i \xi_{p+\xi}$$

In codim 1 (H in TM) this $\sigma_H(P)$ is invertible.

<u>Hörmander's result:</u> If $\sigma_H(P)$ is invertible for $\xi \neq 0$ (then P is hypoelliptic) then P is Fredholm (M for cpct) and

$$\text{Index } P = \int_{T^*M} ch(\sigma_H(P)) \wedge Td(M).$$

<u>proof</u> using $(\frac{1}{2}, \frac{1}{2})$ pseudodiff'l calculus

[The current approach to show P Fredholm:

construct a parametrix Q (a pseudodiff'l op. Q s.t. $PQ-1$, $QP-1$ are smoothing)

[in scalar case: smoothing: $k \in C^\infty(M \times M)$

K is smoothing if $K\varphi(x) = \int k(x,y)\varphi(y)\,dy$

$\Rightarrow$ If φ is distribution on M, $K\varphi$ is smooth.]

We use a different approach: (more elementary actually)

Prove "a priori" estimates for P:

<u>proposition</u>: P diff'l op. on M of H-order d.

(defined earlier w.r.t. (M, H))

Let $\sigma_H(P)$ be invertible.

Then, $\forall A \exists C \mid \forall u$,

$$\|Au\|_2 \leq C\left(\|Pu\|_2 + \|u\|_2\right)$$

where A is a ψdo of H-order $\leq$ Horder(P), C depends on A and u is any smooth func on M. (in case of VB's u is smooth section)