Erik Van Erp / Subelliptic operators / 10/2/8

$P : C^\infty(E) \to C^\infty(F)$

diff'l op. E, F smooth VB's

$\sigma_H(P) =$ principal symbol

on (M, H)

foliated w/

transverse VF's order 2

and parallel VF's

order 1

Theorem If $\sigma_H(P)(x, \xi)$

is invertible, then

$$\|Au\|_{L^2} \le C\left(\|Pu\|_{L^2} + \|u\|_{L^2}\right)$$

(logic: $\forall A$ $\mathrm{order}_H(A) \le \mathrm{order}_H(P)$

$\exists C$ $\forall u \in C^\infty$)

Pf Fourier thy + local approx.

by const. coef. operator. + compactness (P.O.U)

The above is called an "a priori estimate"

$\Downarrow$

P is Fredholm, hypoelliptic, subelliptic

Note: subelliptic means:

Sobolev spaces

$W^d(M)$ = all distributions on M up to order$_H$ d are in $L^2(M)$.

$\uparrow$ Hilbert spc.

locally $\|u\|_{W^d} = \sum_{\|\alpha\| \le d} \|\partial^\alpha u\|^2$ in terms of a chosen basis.

So we can reproduce the identity:

$$\|u\|_{W^d} \le C(\|Pu\|_{L^2} + \|u\|_{L^2}) \quad \forall u \in C^\infty$$

Note: The reverse inequality is trivially true:

$\sqrt{\|Pu\|^2 + \|u\|^2}$ is the graph norm and is the same as the $\| \ \|_{W^d}$ - norm $\Rightarrow$ domain of $\bar{P} \subseteq W^d$.

$\bar{P}$ is Fredholm:

(note: $\overline{P^*} = \bar{P}^*$)

$$\|u\|^2_{W^d} \le C(\|\bar{P}u\|^2 + \|u\|^2)$$

$$\le C(\langle \bar{P}u, \bar{P}u\rangle + \langle u, u\rangle)$$

$$\le C(\langle \bar{P}^*\bar{P}u, u\rangle + \langle u, u\rangle)$$

$$\le C\langle(\bar{P}^*\bar{P}+1)u, u\rangle$$

($\bar{P}^*\bar{P}$ is self-adjoint in the sense of unbdd. op thy. $\bar{P}^*\bar{P} \ge 0$, $\exists$ functional calc.) Spectral Theory $\Longrightarrow$

$$\le C\langle(\bar{P}^*\bar{P}+1)^{\frac{1}{2}}u, (\bar{P}^*\bar{P}+1)^{\frac{1}{2}}u\rangle$$

$$\|u\|_{W^d} \le C\|(\bar{P}^*\bar{P}+1)^{\frac{1}{2}}u\|_{L^2}$$

$$\|(\bar{P}^*\bar{P}+1)^{-\frac{1}{2}}u\|_{W^d} \le C\|u\|_{L^2}$$

so, $(\bar{P}^*\bar{P}+1)^{-\frac{1}{2}} : L^2(M) \to W^d(M) \overset{i}{\hookrightarrow} L^2(M)$ is a bdd. operator.

Rellich Lemma

Inclusion i of Rellich lemma is a compact operator (M cpct.)

$$\Longrightarrow \quad (\bar{P}^*\bar{P}+1)^{-1/2} \text{ is a cpct. op on } L^2(M).$$

and hence is $(\bar{P}^*\bar{P}+1)^{-1}$ is cpct. op. on $L^2(M)$.

To get Fredholmness, $P^t \Rightarrow (\overline{P^t}^*\overline{P^t}+1)^{-1}$ is cpct

need: $\overline{P^t} = P^*$ (tricky)

key ingredient: u, v distributions, if $Pu = v$ and $v \in L^2$ then $u \in W^d$, and $\bar{P}u = v$

$\Rightarrow (\bar{P}\bar{P}^* + 1)^{-1}$ is compact.

Alltogether $\Longrightarrow$ P Fredholm so,

$$D = \begin{pmatrix} 0 & P^t \\ P & 0 \end{pmatrix} \quad \text{on} \quad C^\infty(E \oplus F)$$

$$\bar{D}^* = \bar{D}, \quad (D^2+1)^{-1} \text{ cpct.}$$

Now we can prove an index theorem.