Curvature and the Fundamental Group

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1. Finitely-generated groups and growth functions

Consider a finitely generated group $G = \langle s_1, ..., s_n | r_1, r_2, ...\rangle$, with generators s_i and relations r_i. If there are finitely many relations, then we say that G is finitely presented. For a group element g, we let $l(g)$ denote the minimal length of g as a word in the generators s_i. The *growth function* of G is $\gamma(s)$ and denotes the number of distinct group elements $g \in G$ that satisfy $l(g) \leq s$.

Example 1.

$G = \mathbf{Z} = \langle 1 \rangle$

Then $\gamma(0) = 1$, $\gamma(1) = 3$ and in general $\gamma(s) = 2s + 1$.

We say that γ has *polynomial growth* of degree n if there are constants C_1, C_2 such that $C_1 s^n \leq \gamma(s) \leq C_2 s^n$ for all large enough s. In this example, we can conclude that $\mathbf{Z} = \langle 1 \rangle$ has polynomial growth of degree 1.

Example 2.

Let $G = \langle a, b | a^2 = b^3, ab = ba\rangle = \{a^m b^n : m \in \mathbf{Z}, n = 0, 1, 2\}$

Then $l(a^m b^n) = |m| + n$, and we can compute that $\gamma(s) = 6s - 3$; it also has polynomial growth of degree 1. In fact, G is just another presentation of $\mathbf{Z}$ (the map $a^m b^n \mapsto 3m + 2n$ is an isomorphism). While the γ function depends on the generating set, its asymptotics do not, so we can talk about the growth of a finitely generated group. A celebrated theorem of Gromov states that a finitely generated group has polynomial growth iff it is virtually nilpotent.

Example 3.

Let $F_n = \langle s_1, ..., s_n\rangle$ be the free group on n generators, $n \geq 2$. For any $k \geq 1$ there are $4 \cdot (2n-1)^{k-1}$ words of length k, so $\gamma(s) = 1 + 2\frac{(2n-1)^s - 1}{n+1}$, which has exponential growth. We say that the function γ has *exponential growth* if $\limsup_{s\to\infty} \gamma(s)^{1/s} > 1$

Remark. There exist groups of "intermediate growth", that is superpolynomial but subexponential; for example, the Grigorchuk group. Whether these groups can be fundamental groups of manifolds remains to be seen...

Example 4.

Consider the integer Heisenberg group H consisting of matrices of the form $\begin{pmatrix} 1 & n & p \\ 0 & 1 & m \\ 0 & 0 & 1 \end{pmatrix}$ with $m, n, p \in \mathbf{Z}$.

Another presentation is $H = \langle a, b, c | ac = ca, bc = cb, bab^{-1}a^{-1} = c \rangle$, if we identify the matrix above with $a^m b^n c^p$.

The rules for right multiplication can be written as:

$(a^m b^n c^p)a = a^{m+1} b^n c^{p+n}, (a^m b^n c^p)b = a^m b^{n+1} c^p, (a^m b^n c^p)c = a^m b^n c^{p+1}$

We can prove that $c^{p^2} = a^p b^p a^{-p} b^{-p}$; therefore, $l(c^p) \sim \sqrt{|p|}$, where $\sim$ means "is on the order of".

If $|m| \le s, |n| \le s$ and $|p| \le s^2$ then $l(a^m b^n c^p) \le l(a^m) + l(b^n) + l(c^p) \sim 3s$, so there are on the order of s^4 words of length s. Conversely, if $l(a^m b^n c^p) \le s$ then $|m| \le s, |n| \le s$ and $|p| \le s^2$, so $\gamma(s) \le (2s+1)^4$. Therefore, H has polynomial growth of degree 4.

2. What does curvature have to do with groups?

Classical theorems for a complete manifold M^n:

Bonnet-Myers: If Ric $\ge \delta > 0$ then M is compact and $\pi_1(M)$ is finite.

Preissman: If the sectional curvature satisfies $K < 0$ then every abelian group of $\pi_1(M)$ is infinite cyclic.

Morse theory: if M is compact then it can be given the structure of a CW-complex with finitely many cells in every dimension. Therefore, $\pi_1(M)$ is finitely-presented.

Theorem (Milnor 1968). If Ric ≥ 0 then the growth function $\gamma(s)$ of any finitely-generated subgroup of $\pi_1(M)$ satisfies $\gamma(s) \le k \cdot s^n$ for some constant k.

Proof: Let $\tilde{M}$ be the universal cover of M, with the covering metric. We identify $\pi_1(M)$ with the group of covering transformations of $\tilde{M}$ over M.

Let $H \subset \pi_1(M)$ be a subgroup generated by elements $g_1, ..., g_p$. Fix some $x_0 \in \tilde{M}$ and let $\mu = \max d(x_0, g_i(x_0))$.

We let $N_r(x_0)$ denote the closed ball of radius r around x_0. If $l(g) \le s$

for some $g \in H$ then $d(x_0, g(x_0)) \leq s\mu$, so $g(x_0) \in N_{s\mu}(x_0)$.

As the g's are deck transformations, we can choose some $\epsilon > 0$ small enough so that $N_\epsilon(x_0)$ is disjoint from all its translates $g(N_\epsilon(x_0))$. Then $\gamma(s) Vol N_\epsilon(x_0) \leq Vol N_{s\mu+\epsilon}(x_0)$.

But Ric $\geq 0 \Rightarrow$ by Bishop's theorem, $Vol N_r(x_0) \leq \omega_n r^n$, where ω_n is the volume of the unit ball in Euclidean n-space. On the other hand, $Vol N_\epsilon(x_0)$ is some fixed number, as ϵ is fixed. Therefore,
$\gamma(s) \leq \frac{1}{Vol N_\epsilon(x_0)} Vol N_{s\mu+\epsilon}(x_0) \leq \frac{1}{Vol N_\epsilon(x_0)} \cdot \omega_n (s\mu + \epsilon)^n \leq k \cdot s^n$ for some constant k. □

Theorem (Milnor 1968). If M is compact and $K < 0$ then the growth function of $\pi_1(M)$ satisfies $\gamma(s) \geq a^s$ for some constant $a > 1$.

Remark: This is the maximum growth possible. We can always put an exponential upper bound on $\gamma(s)$, because $\gamma(s+t) \leq \gamma(s)\gamma(t)$ implies $\gamma(s) \leq \gamma(1)^s$. We say that $\gamma(s)$ has exponential growth if $\limsup_{s \to \infty} \gamma(s)^{1/s} > 1$. The theorem says that $\gamma(s)$ has exponential growth.

Another remark: Grigorchuk showed that there are groups with "intermediate" growth, that is subexponential but not polynomial.

Application.

Consider the Heisenberg group G consisting of matrices of the form $\begin{pmatrix} 1 & n & p \\ 0 & 1 & m \\ 0 & 0 & 1 \end{pmatrix}$ with $m, n, p \in \mathbf{R}$. If $m, n, p \in \mathbf{Z}$ then we recover the group H from section 1, and we can conclude that H is a Lie subgroup of G. Then G/H is a 3-manifold with fundamental group H.

As G/H has dimension 3, but its fundamental group has polynomial growth of degree 4, we can conclude that G/H does not admit a metric with Ric ≥ 0. On the other hand, G/H also does not admit a metric with $K < 0$, either.

3. The Milnor Conjecture

Conjecture. Let M be a complete manifold with Ric ≥ 0. Then the fundamental group of M is finitely generated.

Notice that there are manifolds of any dimension ≥ 2 with fundamental groups that are not finitely generated: take the product of the infinitely-holed 2-torus with R^{n-2} to obtain such an n-manifold.

Some work and ideas related to this conjecture:

- Cheeger and Gromoll, the Soul Theorem: if the sectional curvatures satisfy $K \geq 0$ then M is diffeomorphic to the normal bundle over a compact soul S, so in particular $\pi_1(M) = \pi_1(S)$ is finitely generated.

- Wilking: if there is a counterexample to the Milnor conjecture, then it has a covering space with an abelian fundamental grou with is also infinitely generated.

- Schoen and Yau: if M^3 has Ric > 0 then M^3 is diffeomorphic to $\mathbf{R}^3$.

- Li: If M has Euclidean volume growth: $\liminf_{r\to\infty} \frac{VolB_p(r)}{r^n} > 0$ then $\pi_1(M)$ is finite.

- Anderson: $b_1(M) \geq k$ and $\limsup_{r\to\infty} \frac{VolB_p(r)}{r^n} > 0$ then $\pi_1(M)$ is finitely-generated.

- Sormani: if M has small diameter growth: $\limsup_{r\to\infty} \frac{diam\partial B_p(r)}{r} < s_n$ for some constant s_n then $\pi_1(M)$ is finitely generated.

- Wylie: if M has small diameter growth then $\pi_1(M)$ is finitely-presented.

- possible counterexample: the dyadic solenoid complement!

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