Does Curvature Determine the Metric?

Talk by Alina Badus, on a paper by Ravindra Kulkarni

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1 The problem and previous results

Theorema Egregium and others: curvature is an invariant of the metric. Conversely: how far does curvature determine the metric? If a map $f : M \to \bar{M}$ preserves the curvature, is it an isometry? In fact, this encompasses two questions, depending on whether "curvature" means the curvature tensor or sectional curvature.

Myers-Rinow (1935): for complete, simply-connected analytic Riemannian manifolds, if some neighborhood in M is isometric to some neighborhood in $\bar{M}$ then M and $\bar{M}$ are isometric. So the analytic Riemannian manifolds are very special in the class of C^∞ Riemannian manifolds: putting any dent in an analytic one makes it non-analytic.

Cartan (1951): if a "pre-isometry" (constructed in a very special way) preserves the sectional curvature then it is a local isometry.

Ambrose (1956): Let $M, \bar{M}$ be complete, simply-connected manifolds. Fix $p \in M$ and $q \in \bar{M}$ and consider the singly-broken geodesics emanating from p. They correspond (via an isometry $T_pM \to T_p\bar{M}$) to singly broken geodesics emanating from q. If sectional curvature translates along these singly broken geodesics, then the manifolds are globally isometric.

More generally: if a diffeomorphism preserves curvature, must it be an isometry?

NO - in dim 2: the capped cylinder (not analytic mfd). For any 2d mfd, take any diffeomorphism preserving the level curves of $K : M \to \mathbf{R}$. In any dimension: flat tori!

But: in dimensions 4 and higher, this is all that can go wrong.

2 The Main Theorem

isocurved manifolds M, $\bar{M}$:= there is a sectional curvature preserving diffeomorphism $f : M \to \bar{M}$, i.e. $K(\sigma) = \bar{K}(f_*\sigma)$ for all $p \in M$ and all 2-planes $\sigma \subset T_pM$.

Thm (Kulkarni): If $dim \geq 4$ then isocurved manifolds with analytic metrics are globally isometric, except in the case of diffeomorphic, non-globally isometric manifolds with the same constant curvature.

Proof outline: Let $f : M \to \bar{M}$ be a sectional curvature preserving diffeomorphism.

isotropic point $p \in M$:= $K(p, \sigma)$ does not depend on σ.

Claim: Under the hypotheses of the main theorem, the set of non-isotropic points is dense.

Indeed, recall Schur's thm: If M^n is connected, isotropic and $n \geq 3$ then M has constant sectional curvature. By hypothesis, M does not have constant sectional curvature, so there exists at least one non-isotropic point. Suppose the set of non-isotropic points has interior; then by Schur, the sectional curvature must be constant on every component of this interior. But the metric is analytic, so constant curvature on an open set $\Rightarrow$ constant curvature everywhere; contradiction, and the claim is proved.

Claim: f is an isometry on the closure of the set of non-isotropic points.

3 Curvature-preserving implies conformal

curvature tensor R on a vector space V := a bilinear map $R : V \times V \to \text{End}\, V$ which satisfies the identities:

$R(X, Y) + R(Y, X) = 0$

$R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = 0$

$\langle R(X, Y)Z, W\rangle = \langle R(Z, W)X, Y\rangle$

sectional curvature := $K(X, Y) = \frac{|R(X,Y)X,Y)}{\langle X,X\rangle\langle Y,Y\rangle - \langle X,Y\rangle^2}$

Theorem 1. Let $V, \bar{V}$ be two real vector spaces of dimension $n \geq 3$ endowed with inner products $g, \bar{g}$ respectively. Let $R : V \times V \to \text{End}\, V$ and $\bar{R} : \bar{V} \times \bar{V} \to \text{End}\, \bar{V}$ be two curvature tensors and let $K, \bar{K}$ be corresponding sectional curvatures.

Suppose $K \neq$ constant and $f : V \to \bar{V}$ is a sectional curvature preserving linear isomorphism. Then f is a homothety, i.e. $g(f(x), f(y)) = c \cdot g(x, y)$ for some constant c. (follows immediately that $c > 0$).

Proof sketch

frame := an orthonormal basis of V.

acceptable frame := for any triple i, j, k of distinct indices we have $K(e_i, e_j)$, $K(e_j, e_k)$ and $K(E_i, e_k)$ are all distinct.

Lemma 1. K constant iff $R(X, Y)Z = K(\langle X, Z\rangle Y - \langle Y, Z\rangle Z)$.

Proof: doCarmo p. 96. RHS also has the symmetries of R and sectional curvatures coincide.

Lemma 2. If K is not constant then V has an acceptable frame.

Proof: If dim $V = 3$, let $\{e_1, e_2, e_3\}$ be a frame; we can make small adjustments.

If dim $V > 3$, again we can make small adjustments. A little more work to show that the adjustments do not cancel each other.

Why we cannot expect all sectional curvatures to be constant: there are $\frac{n(n-1)}{2}$ sectional curvatures, but the space of curvature-like tensors has dimension $\frac{n^2(n^2-1)}{12}$. In dimensions 4 and higher, there are not enough sectional curvature constraints.

Back to the proof of theorem 1:

Let $\{e_1, ..., e_n\}$ be an acceptable frame. Set $f(e_i) = \bar{e_i}$ and $\bar{g}(\bar{e_i}, \bar{e_j}) = a_i j$.

$K(e_i, e_j) = \bar{K}(\bar{e_i}, \bar{e_j}) \Rightarrow \bar{R}_{ijij} = (a_{ii}a_{jj} - a_{ij}^2)$. Use the symmetries of the curvature to show that $\{\bar{e_1}, ..., \bar{e_n}\}$ are all orthogonal and have the same length.

Corollary. Let $f : M \to \bar{M}$ be a sectional curvature preserving diffeomorphism, and dim≥ 3. Then f is conformal on the closure of the set of non-isotropic points.

conformal map $f :=$ f preserves (non-oriented) angles.

Why this is useful: facts about conformal metrics

conformal metrics g and $\bar{g}$:= there exists a positive function $\varphi : M \to \mathbf{R}$ such that $g(X, Y)_p = \varphi(p)\bar{g}(X, Y)_p$ for all $X, Y \in T_pM$.

Computation gives various relations between connections, curvature tensors, Ricci curvature, scalar curvature. Conclusion: the following quantity is a conformal invariant:

$C(X,Y)Z := R(X,Y)Z + \frac{1}{n-2}\{\mathrm{Ric}(Y,Z)X - \mathrm{Ric}(X,Z)Y + \langle Y,Z\rangle \mathrm{Ric}_0 X - \langle X,Z\rangle \mathrm{Ric}_0 Y\} - \frac{Sc}{(n-1)(n-2)}\{\langle Y,Z\rangle X - \langle X,Z\rangle Y\}$

$C(X,Y)Z :=$ *the Weyl conformal curvature tensor*

$\mathrm{Ric}(X,Y)Z = trace\{Y \to R(X,Y)Z\}$

Ric_0 is an endomorphism of TM defined by: $TM \xrightarrow{c} T^*M \xrightarrow{i} TM$ where $c =$ the canonical map defined by Ric and i is the identification of T^*M with TM via the metric.

Thm 2 (Weyl). If dim $M \geq 4$, M is conformally flat iff $C \equiv 0$.

A manifold M is *conformally flat* if every point of M has a neighborhood on which the metric is $g = \varphi g_0$, where g_0 is the flat metric and φ is a positive real valued function on the neighborhood.

On the other hand, we have:

Proposition. Assume a vector space V is equipped with two inned products g and $\bar{g}$ and two curvature tensors $R, \bar{R}$ such that $\bar{g} = \mu g$ for some

$\mu \in \mathbf{R} > 0$ and $\bar{K} = K$ (equality of corresponding sectional curvatures. Then $\bar{R} = \mu^2 R$, $\bar{\text{Ric}} = \mu^2 \text{Ric}$, $\bar{\text{Sc}} = \mu^2 \text{Sc}$ and $\bar{C} = \mu^2 C$.

4 The non-conformally flat case

Theorem 3. Let (M, g) and $(\bar{M}, \bar{G})$ be isocurved manifolds. Suppose dim $M \geq 4$ and M not conformally flat. Then a curvature preserving diffeomorphism is an isometry.

Proof: M has 2 metrics, g and $f^*\bar{g}$; call this latter $\bar{g}$. Also, the identity map $(M, g) \to (M\bar{g})$ is curvature-preserving.

M not conformally flat, dim $\geq 4 \Rightarrow C \neq 0$ on an open dense subset. But $C = 0$ at isotropic points, so the set of non-isotropic points must be dense. By thm 1 and continuity, $g = \varphi\bar{g}$ for some positive function $\varphi \Rightarrow$ (after some computation) $\bar{C} = \varphi C$. But C is a conformal invariant, and $C \neq 0$, so we must have $\varphi \equiv 1$.

5 The conformally flat case

Theorem 4. Let (M, g) and $(\bar{M}, \bar{G})$ be isocurved manifolds. Suppose dim $M \geq 4$ and M conformally flat. Moreover, assume that the set of non-isotropic points is dense. Then the curvature preserving diffeomorphism is an isometry.

Proof: a lot more computation, since we don't have the Weyl tensor properties.

6 Conclusion - dimension 3, subsequent work

Thm 1 still holds: a sectional curvature preserving diffeomorphism is conformal on the closure of the set of non-isotropic points.

Theorem 6. Let $f : M \to \bar{M}$ be a curvature preserving diffeomorphism. Suppose that M is compact, dim $M \geq 3$, the set of non-isotropic points is dense and for every $p \in M, K(\sigma) < 0$ for some $\sigma \subset T_pM$. Then f is an isometry.

Yau, 1974: counter-example for open 3-manifolds (construct a metric on $\mathbf{R}^3$). But: if $M, \bar{M}$ are nowhere constantly curved compact 3-manifolds then any curvature preserving diffeomorphism is an isometry. *nowhere constantly curved manifold* := the sectional curvature function is not constant at each point (no isotropic points?).

7 Bibliography

Berger, M., *A Panoramic View of Riemannian Geometry*, Springer 2003.
Kulkarni, R.S., *Curvature and Metric*, Ann. of Math., **91**(1970), 311-331.